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Introduction

1.1 Why This Study?

The Riemann hypothesis (RH) has been called the greatest unsolved problem in number theory, and even the greatest unsolved problem in mathematics. It has been around for over 150 years, and is "not expected to be solved any time soon", according to the late Atle Selberg. It has given rise to a great industry of consequences, generalizations, unsuccessful proof attempts, mathematical theories and equivalent forms. It is a class of equivalent forms that is the subject of this volume, namely the classical analytic equivalents.

Leading mathematicians have written a great deal about and around the Riemann hypothesis and its variants and connections, and the interested reader is directed to this valuable, content-rich, set of sources, which includes: 2000–2001, the Clay Mathematics Institute Millennium Prize official problem description, "Problems of the millennium: the Riemann hypothesis" by Enrico Bombieri [34]; 2003, "The Riemann hypothesis" by J. Brian Conrey [57]; 2004, "Problems of the millennium: the Riemann hypothesis" by Peter Sarnak [214]; 2009, "What is the parity phenomenon?" by John Friedlander and Henryk Iwaniec [88]; 2010, "The classical theory of zeta and *L*-functions" by Enrico Bombieri [36]; and 2015, "An essay on the Riemann hypothesis" by Alain Connes [55]. Readers are also encouraged to consult the text "The Riemann hypothesis" by Barry Mazur and William Stein [167, parts II–IV].

These works describe the nature of the hypothesis, its importance and context, and many aspects of current ideas on how it and its relatives might be resolved. No attempt is being made to summarize this material here, but there are some brief comments in the Epilogue to this volume.

A range of arithmetic equivalents to the classical RH are set out in Volume One [39]. That work focuses on many of the classical equivalents to RH. Discussion of modern proposed equivalents, which have geometric and topological ingredients, broadly interpreted, are outside of the scope of this volume, and are in many cases still evolving or conjectural. They represent a great deal of current work.

An equivalence to RH is a very strong implication, so strong that it uses the full power of the hypothesis, and can only be true if the hypothesis is also true. If RH is proved to be true, then each of the equivalences and all of their derived implications of course are true also. If RH is false, then the negation of each of the equivalences is true.

The idea underlying the writing of this volume is that a mathematician who is considering RH, especially a young mathematician starting out in research, would not be expected to have a deep knowledge of wide areas of prerequisite material, but would have special skills and aptitudes for a range of theories and types of mathematical thinking. This volume is intended to enable easy access to quite a wide range of approaches to RH, including series methods, complex variables, Banach and Hilbert spaces, integral equations, measure theory, orthogonal polynomials and cyclotomy, for example.

Leading mathematicians, who have worked and observed the evolution of ideas relating to RH, are in the main optimistic that RH will be proved to be true [210]. These include Enrico Bombieri, Brian Conrey, Henryk Iwaniec and Peter Sarnak. Some, for example Aleksandar Ivić, are waiting for more evidence [215]. Progress with direct methods, for instance enlarging the known zero-free region of $\zeta(s)$, has been very slow. Increasing the height H up to which all zeros with positive imaginary part are on the critical line is not expected to reach near 10^{100}, where variation of the order $\log\log H$ is expected to become quite significant, any time soon.

Because of the litany of unsuccessful attempts to resolve the hypothesis, some believe that it might be undecidable. There is however a simple argument which has been attributed to Turing: If RH is undecidable then it is either false and cannot be proved or true and cannot be proved. But if it is false there is a zero off the critical line, and the existence of such a zero provides an ineffective proof that it is false. Therefore if RH is undecidable it cannot be false, so it must be true but can never be proved. There would not therefore be two forms of mathematics, but a decision to be made: should RH be added as a new axiom?

1.2 Summary of Volume Two

This summary provides an overview of the contents of each chapter of this volume. There is no attempt to be comprehensive, and technical details and formulae, as well as definitions, are absent. These can be found in the introductions and sometimes in the body of the separate chapters.

We begin in Chapter 2 with the oldest criteria, the **Riesz criterion** and the **Hardy–Littlewood criterion**. These take the form of upper bound estimates for sums of series depending on the value of the Riemann zeta function at even natural numbers for Riesz, or odd greater than 1 in the case of Hardy–Littlewood. It took quite a while, in fact over 100 years, for these two closely related criteria to find their proper generalization. This takes the form of the **Báez-Duarte** criterion wherein, from any entire function which satisfies an integrability condition, a modified power series enjoys a particular estimate for real values if and only if RH is true. The chapter includes a summary of work, some of it recent, relating to the so-called Riesz function which underlies the Riesz criterion. It invites extensions to the broader class identified by Báez-Duarte, as steps towards RH.

Two of the most exciting equivalences come from Beurling and his student Nyman, although they were published in the reverse order! These are described in Chapter 3. The **Nyman criterion** is, roughly speaking, the density in the usual Hilbert space based on $(0,\infty)$ of linear combinations of functions based on the fractional part of a real variable – for more precise details see the chapter introduction. Beurling both generalized and simplified Nyman's method, applying it to L^p spaces, and giving a parametrized form of RH, the **Beurling criterion**, again in the form of a subspace density condition.

This might be compared with the Levinson criterion of Section 8.5 or the Salem criterion of Section 8.4 on integral equation equivalences, which is the subject of Chapter 8. The former treats intervals in $(\frac{1}{2},1)$ and the latter lines. Deep understanding of what underlies these three results could provide a path towards RH.

There are many equivalences to RH based on the completed zeta function, $\xi(s)$, or the rotated shifted form, $\Xi(s)$, and some of these are detailed in Chapter 4. Lagarias has two criteria, with one being derived from Robin's inequality [205] as reported in Volume One [39]. The criterion in this volume is that the real part of the logarithmic derivative of $\xi(s)$ should be positive on $(\frac{1}{2},\infty)$. He found a general setting for functions of a similar type to $\xi(s)$, namely those which are entire and have so-called admissible zero sets.

The **Sondow–Dumitrescu criterion** is that RH is equivalent to the modulus of ξ being strictly increasing in σ on each horizontal half line $(\frac{1}{2},\infty)\times\{t\}$.

The most popular and generalized equivalence in this chapter is the **Li criterion**. It takes the form of an infinite set of inequalities, parametrized by natural numbers, summed over the non-trivial zeros of $\zeta(s)$. Lagarias and Bombieri have shown that this criterion is equivalent to that of Weil (reported on in Section 9.5) if one chooses a particular sequence of test functions. This criterion has been extended to Dirichlet L-functions and to functions in the so-called Selberg class, which is very broad.

Pólya worked long and hard on RH, and the de Bruijn–Newman constant of Chapter 5 is based on his ideas. The rotated shifted Riemann xi function, $\Xi(s)$, can be expressed as a complex Fourier transform. A real parameter is introduced into this expression, giving a perturbed form $\Xi_\lambda(s)$, so, of course, RH is equivalent to all of the zeros of $\Xi(s) = \Xi_0(s)$ being real. Related is the de Bruijn–Newman constant Λ, which is such that all zeros of $\Xi_\lambda(s)$ are real if and only if $\lambda \geq \Lambda$. This equivalence is (graced with being) called the **de Bruijn–Newman criterion**.

The chapter reports on progress bounding Λ, with Newman first showing in 1977 that $-\infty < \Lambda$ and Ki et al. in 2009 that $\Lambda < \frac{1}{2}$. An attempt has been made in this volume to simplify the related material of de Bruijn. The result of Newman was not explicit, and recent work does not depend on it, but rather depends on the concept of Lehmer pairs of zeros. Here a computationally tractable version of the method due to Csordas and his coworkers and others is presented. The best current published lower bound for Λ by Saouter, Gourdon and Demichel has $-1.14541 \times 10^{-11} < \Delta$, so if RH is true, this parametrization shows it “only just squeezes through the narrow door of truth”.

The upper bound needs much more work if this method is to successfully show RH is true, as many suspect. The chapter shows that $\Lambda \leq \frac{1}{2}$ is a straightforward deduction.

There are two polynomial criteria, but they are quite different. The **Cardon–Roberts criterion**, as described in Chapter 6, is based on a particular family of orthogonal polynomials using a measure coming from $\Xi(s)$. It uses basic results for orthogonal polynomials, which are derived in the chapter, as well as concepts from the theory of quasi-analytic functions and the Hamburger moment problem, each of which is also described. The criterion is restricted in the sense that it is for RH with simple zeros, and not just RH, so removing the restriction could be a potential development. The criterion is unusual in that it takes the form of an expression of a (normalized) limit of even polynomials from the family being $\Xi(s)$ for every s. There is also a form for the odd polynomials.

Clearly the path to RH would require more detailed knowledge of the particular polynomials, including for example some recursive relationships and an examination of their roots. The method has been extended to automorphic L-functions by Mazhouda and Omar.

Some equivalences to RH are, on the face of it, independent of others, because the ideas are different and they do not use other criteria or it seems imply them, other than through RH. This second polynomial criterion, the **Amoroso criterion**, has a simple structure giving a bound for the height of the product of the first N cyclotomic polynomials. It is not independent, in that, like several other criteria, it uses Littlewood’s bound for sums of the Möbius μ function. There is an accompanying criterion in terms of the value of the derivative of the product at roots of unity.

Like the Báez-Duarte criterion of Chapter 2, Chapter 8 is a good example where further work with an individual statement has produced its underlying structure and a wide family of generalizations. The criteria of Volchkov, Balazard, Saias and Yor of 1995 and that of the last three listed authors of 1999 were generalized in several different ways in 2012, using a fundamental lemma. Here we have given the reader its proof and a sample of its applications, all based on contour integrals of functions of the form of a meromorphic function times the logarithm of a meromorphic function. We call one of these applications the **Sekatskii–Beltraminelli–Merlini criterion** and use it to show how a path towards showing that RH could be false might be found.

As well as these integrals depending rather directly on $\zeta(s)$, there are two integral equation types of criteria which are close in spirit to the Banach and Hilbert space methods of Chapter 3. These have already been mentioned in the context of that chapter and are the **Salem criterion** and the **Levinson criterion**. Even though Levinson believed that the sufficient condition for RH was too easy to be useful, the use of either the sophisticated proof of Wiener's theorem or the Hahn–Banach together with Radon–Nikodym theorems from functional analysis underpinning these results, shows that more attention could well be given to these criteria.

Chapter 9 provides entry-level material on the ground-breaking work of André Weil. It includes Bombieri's derivation of the Weil explicit formula, and a derivation of the **Weil criterion** for RH, a positivity condition based on the explicit formula. The **Bombieri criterion** is proved, together with a summary of the path he mapped out towards RH based on variational methods.

The Weil conjectures for varieties over finite fields are stated, including their form of RH, and the history of their successful resolution detailed. Elliptic curves over finite fields are introduced and, subject to some preliminary properties which are left to the literature, a complete proof of the conjectures in the special case of elliptic curves is given. To complete the chapter, Weil's plan to bridge between number fields and function fields is presented by way of his written commentary on two of his seminal papers. The completing of his programme towards RH has been the subject of intensive research, from the 1950s up to the present day.

The **Verjovsky criterion** of Chapter 10 is based on a parametrized set of measures on sets of continuous functions on the positive reals having compact support. The criterion takes the form of an estimate for the error in the value of the measure as the parameter tends to zero. The primes come in through use of the Euler totient function in the construction of the measures, although few properties of this function are used in the proofs. Even though this criterion was published after that of Zagier and Sarnak outlined in the chapter, its general form is very similar, showing the possible existence of a common underlying abstraction.

Chapter 11 sets out the insightful ideas of Yoshida based on Weil's explicit formula and its underlying functional. Yoshida's approach is based in particular on restricting Weil's functional to functions with support in intervals $[-a,a]$ and deriving an equivalence to RH requiring the condition to hold for all $a > 0$, namely the **Yoshida criterion**. He gives a proof in the case $a = \log\sqrt{2}$. The reader might observe that the methods are far from obvious, and require the explicit calculation of functional coefficients with respect to two different bilinear forms and their completions. In addition, part of the method, as one might expect in any resolution of RH, uses techniques from combinatorial number theory, unlike for example Chapter 3, and many other chapters in this volume, which are purely analytic.

Even though the exposition of Yoshida's ideas does not include the extension to number fields, the original does that, at least in part. In this presentation, the challenging numerical work which was needed for the original has been replaced by something a little more "canonical".

This volume includes three developments arising from the Weil explicit formula: the variational approach to RH of Bombieri in Chapter 9, the account of the resolution of the Weil conjectures also in Chapter 9, and this criterion of Yoshida.

Chapter 12 is somewhat different from many of the others. Firstly it broadens the scope of RH by including Dirichlet L-functions and their Riemann hypothesis, the generalized or general Riemann hypothesis (GRH). Secondly, for this class of L-functions it has little by way of equivalences – indeed only one is included, the **Titchmarsh criterion**. Thirdly it deals with approximations to the hypotheses, namely the Gallagher estimate and the Bombieri–Vinogradov estimate. These unconditional results have found very serious applications in number theory, and sometimes can replace the use of RH or GRH. Indeed, there are several important instances where GRH was first used to prove a result and then successfully removed by using an unconditional estimate. The chapter illustrates this process with examples and concludes with some of the well-known conjectures, such as the non-vanishing of all Dirichlet L-functions on $(0,1)$, which includes in particular the non-existence of the Siegel zeros conjecture. These examples also include a description of recent progress that has been made towards a resolution of the twin primes conjecture.

In Chapter 13 an equivalence is derived which goes to the heart of results in analytic number theory, namely the range of validity of an asymptotic estimate for the count of smooth numbers. This is the **Hildebrand criterion**. The existing best unconditionally known range can be very significantly extended, provided we assume RH, and in turn the existence of such an extension implies RH. Like Robin's inequality and its extensions in Volume One [39], the Hildebrand criterion is a statement not just about the distribution of individual primes, but on how they work together. Ivić has judged this criterion "remarkable". Finding methods to reduce the lower bound for the unconditional range of validity is the stuff of classical analytic number theory and would provide a path to RH.

1.3 How to Read This Book

The separate chapters are, in the main, independent. However, Chapter 6 depends on Chapter 4, and Chapter 11 depends on Chapter 9. Also there are relations between Chapters 3 and 8. The 11 appendices, A through K, are important components. Readers with good undergraduate mathematics backgrounds should be able to find most of the specialized results fully proved. In addition, at the end of the introductory section for each chapter there is listed some hopefully accessible background reading. For more general introductions, see the suggestions in Chapter 1 of Volume One [39]. To this list must be added the valuable text [37].