

## How to Divide When There Isn't Enough

*How to Divide When There Isn't Enough* develops a rigorous yet accessible presentation of the state of the art for the adjudication of conflicting claims and the theory of taxation. It covers all aspects one may wish to know about claims problems: the most important rules, the most important axioms, and how these two sets are related. More generally, it also serves as an introduction to the modern theory of economic design, which in the last twenty years has revolutionized many areas of economics, generating a wide range of applicable allocation rules that have improved people's lives in many ways. In developing the theory, the book employs a variety of techniques that will appeal to both experts and nonexperts. Compiling decades of research into a single framework, William Thomson provides numerous applications that will open a large number of avenues for future research.

William Thomson is the Elmer Milliman Professor of Economics at the University of Rochester. He is the author of several books including *A Guide for the Young Economist*, which has appeared in four translations, and over one hundred articles. In 2001, he won the University Award for Excellence in Graduate Teaching at the University of Rochester. He is a Fellow of the Econometric Society, the Society for Economic Theory, and the Game Theory Society.

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**How to Divide  
When There Isn't Enough**  
From Aristotle, the Talmud, and  
Maimonides to the Axiomatics of  
Resource Allocation

William Thomson  
*University of Rochester*



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## Contents

<i>List of Figures</i>	<i>page</i> xii
<i>List of Tables</i>	xix
<i>Acknowledgments</i>	xx
<i>General Notation</i>	xxi
1 Introduction	1
1.1 Claims Problems	1
1.2 The Model	3
1.3 Two Puzzles in the Talmud	9
1.4 Three Approaches	11
1.4.1 Direct Approach	12
1.4.2 Axiomatic Approach	12
1.4.3 Game-Theoretic Approach	15
1.5 Historical Note	16
1.6 Road Map	16
1.7 How to Use This Book	17
1.8 Concluding Comment	18
2 Inventory of Division Rules	21
2.1 An Inventory of Rules	22
2.1.1 Proportional Rule	22
2.1.2 Constrained Equal Awards Rule	23
2.1.3 Constrained Equal Losses Rule	26
2.1.4 Concede-and-Divide	28
2.1.5 Piniles' Rule	31
2.1.6 Talmud Rule	32
2.1.7 Constrained Egalitarian Rule	34
2.1.8 Random Arrival Rule	37
2.1.9 Minimal Overlap Rule	38
2.1.10 Rule Based on Random Stakes	43

viii	<b>Contents</b>	
	2.2 Families of Rules	45
	2.2.1 Sequential Priority Family	45
	2.2.2 Young's Family	46
	2.2.3 ICI and CIC Families	53
	2.3 Summary	60
3	<b>Basic Properties of Division Rules</b>	62
	3.1 Balance	62
	3.2 Continuity	63
	3.3 Homogeneity	64
	3.4 Lower and Upper Bounds on Awards and Losses	65
	3.4.1 Defining Bounds	65
	3.4.2 Recursive Assignment of Lower Bounds	72
	3.5 Conditional Full Compensation, Conditional Null Compensation, and Related Properties	75
	3.6 Symmetry Properties	79
	3.7 Order Preservation Properties	89
4	<b>Monotonicity Properties</b>	94
	4.1 Endowment Monotonicity and Related Properties	95
	4.2 Claim Monotonicity and Related Properties	105
	4.3 Inverse Sets Axioms	115
5	<b>Claims Truncation Invariance and Minimal Rights First</b>	118
	5.1 Claims Truncation Invariance	119
	5.2 Minimal Rights First	123
6	<b>Composition Down and Composition Up</b>	131
	6.1 Composition Down	131
	6.2 Composition Up	140
7	<b>Duality</b>	157
	7.1 Duality for Rules	157
	7.2 Duality for Properties	165
	7.3 Duality for Theorems	171
	7.4 Characterizations	172
8	<b>Other Invariance Properties</b>	182
	8.1 No Advantageous Transfer	182
	8.2 Claims Separability and Variants	184
	8.3 Convexity and Additivity Properties	187
	8.4 Rationalizing Rules as Maximizers of Binary Relations	195
9	<b>Operators</b>	200
	9.1 Claims Truncation Operator	200



<b>Contents</b>	ix
9.2 Attribution of Minimal Rights Operator	202
9.3 Convexity Operator	205
9.4 Relating and Composing the Operators	206
9.5 Preservation of Properties under Operators	214
9.5.1 Properties Preserved under Claims Truncation	215
9.5.2 Properties Preserved under Attribution of Minimal Rights Operator	218
9.5.3 Properties Preserved under the Composition of the Claims Truncation and Attribution of Minimal Rights Operators	219
9.5.4 Properties Preserved under Convexity	221
9.6 Extension Operators	222
9.7 Summarizing	227
10 Variable-Population Model: Consistency and Related Properties	229
10.1 The Variable-Population Model	230
10.2 Consistency and Related Properties	231
10.3 Converse Consistency	239
10.4 Other Logical Relations between Consistency, Its Converse, and Other Properties	241
10.5 Lifting of Properties by Bilateral Consistency	249
10.6 Characterizations	255
10.7 Average Consistency	266
11 Constructing Consistent Extensions of Two-Claimant Rules	270
11.1 A General Extension Technique	271
11.2 Consistent Extensions of Two-Claimant Rules Satisfying Equal Treatment of Equals	277
11.2.1 Consistent Extension of Weighted Averages of the Two-Claimant Constrained Equal Awards and Constrained Equal Losses Rules	277
11.2.2 Two-Claimant Rules that Have No Consistent Extension	282
11.2.3 Consistent ICI and CIC Rules	283
11.2.4 Other Consistent Families	288
11.3 Consistent Extensions of Two-Claimant Rules that May Not Satisfy Equal Treatment of Equals	289
11.3.1 Generalizing the Talmud Rule by Not Insisting on Equal Treatment of Equals	290
11.3.2 Consistent Extensions of Two-Claimant Rules Satisfying Homogeneity, Composition Down, and Composition Up	297

x	<b>Contents</b>	
	11.4 Further Characterizations Involving Consistency and Other Axioms but Not Equal Treatment of Equals	302
12	Variable-Population Model: Other Properties	308
	12.1 Population Monotonicity and Related Properties	308
	12.2 Guarantee Structures	313
	12.3 Merging and Splitting Claims; Manipulation Issues and Extension Operators	315
	12.3.1 No Advantageous Merging or Splitting and Variants	316
	12.3.2 Extension Operators Based on the Merging of Claims	320
	12.4 Replication and Division: Invariance and Limit Results	324
	12.4.1 Convergence of Rules under Replication	330
	12.5 Balanced Impact and Potential	334
	12.6 Multiple Parameter Changes; Logical Relations and Characterizations	335
13	Ranking Awards Vectors and Ranking Rules	339
	13.1 Orders Based on the Lorenz Criterion	340
	13.1.1 Maximality and Minimality Results	340
	13.1.2 A Criterion for Lorenz-Domination within the ICI Family	342
	13.2 Preservation of Orders by Operators	348
	13.3 Lifting of Orders by Bilateral Consistency	351
	13.4 Other Properties of Rules Pertaining to Orders	352
	13.5 Orders Based on Gap and Variance	354
14	Modeling Claims Problems as Games	359
	14.1 Modeling Claims Problems as Cooperative Games	359
	14.1.1 Bargaining Games	359
	14.1.2 Coalitional Games	367
	14.2 Modeling Claims Problems as Strategic Games	380
	14.2.1 Game of Stakes	380
	14.2.2 Game of Rules	383
	14.2.3 Sequential Game of Offers	388
15	Variants and Generalizations of the Base Model	390
	15.1 Claims Problems in Which No Claim Exceeds the Endowment	390
	15.2 Claims Problems in Which the Data Are Natural Numbers	391
	15.3 Claims Problems with a Large Number of Claimants	393
	15.4 Surplus-Sharing Problems	395
	15.5 Generalizing the Notion of a Rule	396

<b>Contents</b>	xi
15.6 Computational Issues	397
15.7 Incorporating Additional Information into the Model	397
15.8 Experimental Testing	405
15.9 A Concluding Comment	406
16 Summary Graphs and Tables	408
17 Appendices	416
17.1 Deriving a Formula for the Minimal Overlap Rule	416
17.2 More about the CIC Rules	417
17.3 Paths of Awards of the DT Rule	420
17.4 Neither Claim Monotonicity Nor No-Transfer Paradox Is Preserved under the Duality Operator	422
17.5 Claim Monotonicity Is Not Preserved under the Attribution of Minimal Rights Operator	426
17.6 Lifting of Properties by Bilateral Consistency	428
17.7 Characterizing the Family of Equal-Sacrifice Rules	429
17.8 On the Existence and Uniqueness of Average Consistent Extensions	432
17.9 Constructing Consistent Extensions	434
17.10 On the Consistent Members of the CIC Family	436
17.11 Characterizing a Family of Sequential Talmud Rules	438
17.12 Completion of the Proof of the Characterization of Family $\mathcal{M}$	440
17.13 Population Monotonicity Is Not Preserved under Duality	444
17.14 Characterization of the Constrained Equal Awards Rule as Offering Maximal Group Guarantees	447
17.15 Under Replication, the Random Arrival Rule Converges to the Proportional Rule	449
17.16 Convexity of the TU Coalitional Game Associated with a Claims Problem	452
17.17 Proof of the Correspondence between the Talmud Rule and the Nucleolus, and of the Constrained Equal Awards Rule and the Dutta–Ray Solution	453
<i>References</i>	456
<i>Index</i>	472

## Figures

1.1	Identifying the awards vectors of a two-claimant problem	<i>page</i> 4
1.2	Identifying the awards vectors of a three-claimant problem	5
1.3	Three ways of depicting a division rule	8
1.4	Two puzzles in the Talmud	10
2.1	Proportional rule	23
2.2	Constrained equal awards and constrained equal losses rules	25
2.3	A simple way to calculate the constrained equal awards vector for a fixed claims vector and three values of the endowment	26
2.4	Constrained equal awards rule	27
2.5	Calculating the constrained equal losses vector for a fixed claims vector and three values of the endowment	28
2.6	Constrained equal losses rule	29
2.7	Scenario underlying concede-and-divide, a two-claimant rule	30
2.8	Concede-and-divide	30
2.9	Piniles' rule applied to the examples in the Talmud	32
2.10	Comparing Piniles' rule and concede-and-divide for two claimants	33
2.11	The Aumann–Maschler proposal, which simultaneously rationalizes the recommendations made in the Talmud for the contested garment and marriage contract problems	33
2.12	Constrained egalitarian rule applied to two-claimant examples	37
2.13	Constrained egalitarian rule applied to the two claims vectors in the Talmud	37
2.14	Random arrival rule	39
2.15	Configurations of claims yielding minimal overlap	42
2.16	Minimal overlap rule	42
2.17	Two sequential priority rules	46

<b>List of Figures</b>	xiii
2.18 Defining a Young rule	47
2.19 Young representations of the proportional, constrained equal awards, and constrained equal losses rules	49
2.20 Young representation of the Talmud rule when there is a maximal value that a claim can take, $c_{max}$	50
2.21 Young representations of Piniles' and constrained egalitarian rules when there is a maximal value that a claim can take, $c_{max}$	50
2.22 Young representations of the Talmud and Piniles' rules when no upper bound is imposed on claims	52
2.23 Schedules of awards of a four-claimant ICI rule for a particular claims vector	55
2.24 Paths of awards of four two-claimant ICI rules	58
2.25 Reverse Talmud rule	59
2.26 Young representation of the reverse Talmud rule when there is a maximal value that a claim can take, $c_{max}$	60
2.27 Paths of awards of four two-claimant CIC rules	61
3.1 The minimal rights lower bounds for two claimants	66
3.2 The minimal rights lower bounds for three claimants	67
3.3 The $\frac{1}{ N }$ -truncated-claims lower bounds on awards for two claimants	70
3.4 Conditional full compensation	77
3.5 Weighted versions of the proportional, constrained equal awards, and constrained equal losses rules	83
3.6 Weighted versions of concede-and-divide	85
3.7 Weighted versions of the Talmud rule seen as a hybrid of the constrained equal awards and constrained equal losses rules	87
3.8 Two rules violating order preservation and one rule satisfying the property for a particular claims vector	90
3.9 Group order preservation for three claimants	92
4.1 Endowment monotonicity	95
4.2 Progressivity and regressivity	100
4.3 Concavity and convexity	101
4.4 Various notions of visibility from below	103
4.5 Various notions of visibility from above	104
4.6 Claim monotonicity	105
4.7 Two violations of claim monotonicity	106
4.8 Bounded gain from claim increase	107
4.9 Two properties pertaining to changes in claims	109
4.10 For two claimants, claim monotonicity implies no transfer paradox	111

xiv	<b>List of Figures</b>	
4.11	No transfer paradox does not imply claim monotonicity	112
4.12	Two endowment-monotonicity impact properties	113
4.13	Characterizing the constrained equal awards rule	114
4.14	Inverse sets for six rules	116
5.1	Claims truncation invariance	120
5.2	Paths of awards of rules satisfying claims truncation invariance	121
5.3	Illustrating a violation of minimal rights first	124
5.4	Two rules and minimal rights first	125
5.5	Characterizing concede-and-divide	126
5.6	Characterizing the family of weighted concede-and-divide rules	128
5.7	Characterizing a family of nonhomogeneous rules generalizing concede-and-divide	129
6.1	Composition down	132
6.2	Two rules and composition down	133
6.3	The signature of a rule satisfying composition down is a monotone space-filling tree in awards space	135
6.4	If a rule satisfies homogeneity and composition down, its generating curves are visible from the origin except possibly for an initial segment	138
6.5	Monotone path rules are generalizations of the constrained equal awards rule	139
6.6	Composition up	141
6.7	Three rules and composition up	142
6.8	The weighted proportional rules with unequal weights satisfy neither composition down nor composition up	142
6.9	Characterizing the constrained equal awards rule	145
6.10	Characterizing the constrained equal awards rule	148
6.11	Four members of family $\mathcal{D}$	150
6.12	Two members of family $\mathcal{D}$ satisfying anonymity	152
6.13	The paths of awards of a rule satisfying homogeneity, composition down, and composition up are piecewise linear in (at most) two pieces	153
6.14	One rule satisfying composition down and composition up, and a second one satisfying homogeneity in addition	155
7.1	Self-duality	158
7.2	Relating the ICI and CIC families	159
7.3	The random arrival rule is self-dual	160
7.4	Averaging the constrained equal awards and constrained equal losses rules	163

<b>List of Figures</b>	xv	
7.5	Duals of a weighted concede-and-divide, Piniles', and the constrained egalitarian rules	163
7.6	Linked claim-endowment monotonicity	166
7.7	Characterizing the proportional rule: the initial step	173
7.8	Characterizing the proportional rule: iterating	173
7.9	Characterizing the proportional rule: a second proof	175
7.10	Characterizing the constrained equal losses rule	176
7.11	Characterizing concede-and-divide	178
8.1	The constrained equal awards rule violates no advantageous transfer	183
8.2	Two convexity properties	189
8.3	Two additivity properties	190
8.4	Two invariance properties with respect to certain changes in claims vectors	193
8.5	Characterizing concede-and-divide	195
8.6	Two more invariance properties with respect to changes in the claims vector	197
9.1	Proportional rule operated from truncated claims	201
9.2	Constrained equal losses rule operated from truncated claims	202
9.3	Proportional rule operated from minimal rights.	204
9.4	Constrained equal awards rule operated from minimal rights	204
9.5	Three weighted averages of the constrained equal awards and constrained equal losses rules	206
9.6	The duality operator relates the claims truncation and minimal rights operators	207
9.7	Relating the claim truncation and attribution of minimal rights operators through duality: an application	209
9.8	The attribution of minimal rights operator and the claims truncation operator commute	211
9.9	Endowment monotonicity is not preserved under claims truncation, even for two claimants	217
9.10	Neither composition down nor composition up is preserved under convexity	222
9.11	Operator to extend to the entire domain a rule $S$ defined on the domain of problems in which the endowment is at most as large as the largest claim	224
9.12	Operator to extend to the entire domain a rule $S$ defined on the domain of problems in which the endowment is at least as large as the largest claim	226
10.1	Consistency	232
10.2	The minimal overlap rule is not consistent	233

xvi	<b>List of Figures</b>	
10.3	Young's rules are consistent	234
10.4	A generalized Young representation of a monotone path rule	236
10.5	There is at most one bilaterally consistent rule that coincides for two claimants with a prespecified endowment monotonic rule	244
10.6	Elevator Lemma	245
10.7	Lifting of claim monotonicity with the assistance of endowment monotonicity	254
10.8	Characterizing the proportional rule	264
10.9	Average consistency	267
11.1	Constructing the consistent extension of a strictly endowment monotonic two-claimant rule	273
11.2	Constructing the consistent extension, if it exists, of an endowment monotonic (but not strictly so) two-claimant rule	274
11.3	Constructing the consistent extension, if it exists, of an endowment monotonic (but not strictly so) two-claimant rule (continued)	275
11.4	Constructing the consistent extension of a two-claimant weighted constrained equal awards rule	276
11.5	Constructing the consistent extension of concede-and-divide	277
11.6	Proof of Theorem 11.1: constructing the paths of awards for $c_{\{1,2\}}$ , $c_{\{1,3\}}$ , and $c_{\{2,3\}}$	279
11.7	Proof of Theorem 11.1: constructing the first segment of $\Pi$	280
11.8	Proof of Theorem 11.1: constructing the first segment of $\Pi$ (continued)	281
11.9	Young representation of the ICI* rule associated with some $\gamma \in \Gamma$ , $ICI^\gamma$	284
11.10	Proof of Theorem 11.5	286
11.11	A member of family $\mathcal{T}^1$	291
11.12	Proof of Lemma 11.2(i); construction of $\Pi$	294
11.13	Proof of Lemma 11.2(ii); relating the claimants' weights	295
11.14	A member of family $\mathcal{M}$	298
11.15	Lemma 11.5: Partitioning the set of potential claimants into priority classes	300
12.1	Population monotonicity	309
12.2	Who's who in a replica problem	325
12.3	Showing that anonymity and converse consistency together imply invariance under replication for two-claimant problems	328
12.4	Convergence of the minimal overlap rule to the constrained equal losses rule under replication of a two-claimant example	331



<b>List of Figures</b>	xvii
13.1 A criterion for Lorenz domination within the ICI family	344
13.2 Passing from one ICI rule to the other	346
13.3 Characterizing the constrained equal awards rule as gap minimizer	355
13.4 Characterizing the constrained equal awards rule as variance minimizer	355
13.5 Summarizing Lorenz rankings	357
14.1 Claims problems and their associated bargaining games, and two bargaining solutions	362
14.2 The extended equal losses bargaining solution	363
14.3 What it means for a division rule to correspond to a solution to bargaining games	364
14.4 Generating the paths of awards of the equal area rule	365
14.5 Core of the coalitional game associated with a claims problem	371
14.6 What it means for a division rule to correspond to a solution to coalitional games	373
14.7 Commutative diagram	375
14.8 Game in which strategies are claims on specific parts of the endowment	381
14.9 A game of rules for two claimants	384
14.10 A dual game of rules	387
15.1 Two models related to the base model of claims resolution	391
15.2 Two rules for surplus-sharing problems	396
15.3 Nontransferable utility claims problems	398
16.1 Paths of awards for six rules in the two-claimant case	409
16.2 Paths of awards for six more rules in the two-claimant case	410
17.1 Essential uniqueness of the configuration of claims yielding minimal overlap	417
17.2 Schedules of awards of a four-claimant CIC rule for a particular claims vector	419
17.3 Paths of awards of the DT rule	421
17.4 Paths of awards of the DT rule (continued)	422
17.5 Claim monotonicity is not preserved under duality	423
17.6 Rule 17.1 is claim monotonic	424
17.7 No transfer paradox is not preserved under duality	425
17.8 Claims monotonicity is not preserved under attribution of minimal rights	427
17.9 Paths of awards of Rule 17.4	428
17.10 Completing the proof of Lemma 10.14a	430
17.11 Characterizing the family of equal-sacrifice rules	431

xviii     **List of Figures**

17.12	Proof of Theorem 11.3	435
17.13	Young representations of two CIC* rules	437
17.14	Lemma 11.7, Substep 1-1	440
17.15	Lemma 11.7, Substep 1-2	441
17.16	Lemma 11.7, Step 2, Case 1	442
17.17	Lemma 11.8	444
17.18	Population monotonicity is not preserved under duality	446

## Tables

2.1	Random arrival scenario applied to a marriage contract problem in the Talmud	<i>page 38</i>
16.1	Showing which of the main properties the main rules satisfy	411
16.2	Showing which properties are preserved under the operators	414

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## General Notation

Set of natural numbers	$\mathbb{N}$
Set of real numbers	$\mathbb{R}$
The closed interval in $\mathbb{R}$ with endpoints $a$ and $b$	$[a, b]$
The open interval with endpoints $a$ and $b$	$]a, b[$
Given $x, y \in \mathbb{R}$ , for each $i \in N$ , $x_i \leq y_i$	$x \leq y$
$x \leq y$ and $x \neq y$	$x \leq y$
For each $i \in N$ , $x_i < y_i$	$x < y$
Vector $x$ from which $i$ th coordinate has been deleted	$x_{-i}$
Vector $x$ in which $i$ th coordinate has been replaced by $x'_i$	$(x'_i, x_{-i})$
Vector $x$ with coordinates rewritten in increasing order	$\tilde{x}$
Interior of $A \subset \mathbb{R}^\ell$	$\text{int}\{A\}$
Interior of $A \subset \mathbb{R}^\ell$ relative to $\mathbb{R}_+^\ell$	$\text{rel.int}\{A\}$
Sets of claimants	$N, N', \bar{N}, \dots$
Generic claims vectors	$c, c', \bar{c}, \dots$
Generic endowments	$E, E', \bar{E}, \dots$
Generic claims problems	$(c, E), (c', E'), (\bar{c}, \bar{E}), \dots$
Domain of problems with claimant set $N$	$\mathcal{C}^N$
Awards space for claimant set $N$	$\mathbb{R}_+^N$
Set of awards vectors of $(c, E) \in \mathcal{C}^N$	$X(c, E) \equiv \{x \in \mathbb{R}^N : 0 \leq x \leq c, \sum x_i = E\}$
Claimant $i$ 's claim $c_i$ truncated at $E$	$t(c_i, E)$
Vector of claims $c$ each truncated at $E$	$(t(c_i, E))_{i \in N} = t(c, E)$
Cardinality of the set $A$	$ A $
Family of finite subsets of $\mathbb{N}$	$\mathcal{N}$
Union $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$	$\mathcal{C}$
Class of strict orders on $N$	$\mathcal{O}^N$
Class of weak orders on $N$	$\tilde{\mathcal{O}}^N$

xxii      **General Notation**

Class of bijections on $N$	$\Pi^N$
$i$ th unit vector in $\mathbb{R}^N$	$e_i$
Unit simplex in $\mathbb{R}^N$	$\Delta^N \equiv \{x \in \mathbb{R}_+ : \sum x_i = 1\}$
Vector of equal coordinates in $\Delta^N$	$e_N$
Given $N' \subset N$ , projection of $x \in \mathbb{R}^N$ onto $\mathbb{R}^{N'}$	$x_{N'}$
Segment connecting $x$ and $y \in \mathbb{R}^N$	$\text{seg}[x, y]$
Broken segment connecting $x^1, \dots, x^k \in \mathbb{R}^N$	$\text{bro.seg}[x^1, \dots, x^k]$
Given $x, y \in \mathbb{R}^N$ such that $x \leq y$ , set of vectors $z$ such that $x \leq z \leq y$	$\text{box}[x, y]$

**Notation for Division Rules**

Generic rules	$S, S', \bar{S} \dots$
Path of awards of $S$ for $c$	$p^S(c)$

**Individual Rules**

Proportional rule	$P$
Concede-and-divide (for $ N  = 2$ )	$CD$
Reverse concede-and-divide (for $ N  = 2$ )	$CD^r$
Constrained equal awards rule	$CEA$
Constrained equal losses rule	$CEL$
Talmud rule	$T$
Reverse Talmud rule	$T^r$
Piniles' rule	$Pin$
Constrained egalitarian rule	$CE$
Random arrival rule	$RA$
Minimal overlap rule	$MO$
Random stakes rule	$RS$
Adjusted proportional rule	$AP$
Average of $CEA$ and $CEL$	$Av$

**Families of Rules**

Sequential priority rule relative to order $\prec \in \mathcal{O}^N$	$SP^{\prec}$
Sequential Talmud rule relative to order $\preceq \in \tilde{\mathcal{O}}^N$ and weights $w \in \Delta^N$	$ST^{\preceq, w}$
Young rule of representation $f \in \Phi$	$Y^f$
Equal sacrifice rule relative to $u \in \mathcal{U}$	$ES^u$
ICI rule relative to $H \in \mathcal{H}^N$	$ICI^H$
CIC rule relative to $H \in \tilde{\mathcal{H}}^N$	$CIC^H$

### General Notation

xxiii

TAL rule of parameter  $\theta \in \Delta^N$   $T^\theta$   
 Reverse TAL rule of parameter  $\theta \in \Delta^N$   $U^\theta$

### Operating on Rules

Rule  $S$  subjected to the  
 attribution of minimal rights operator  $S^m$   
 claims truncation operator  $S^t$   
 duality operator  $S^d$   
 operator  $p$  and then to operator  $p'$   $S^{p' \circ p}$

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