CHAPTER 1

Introduction

1.1 CLAIMS PROBLEMS

How to divide when there isn't enough? When a group of agents has claims on a resource that add up to more than what is available, how should the resource be divided? A "division rule," or "rule" for short, associates with each such "claims problem" a division among the claimants of the amount available. Our goal is to survey the literature devoted to identifying the most desirable rules.

A primary concern of this literature is with a firm going bankrupt, its liquidation value having to be apportioned among its creditors. The model we study here, however, can be given many other interesting interpretations; covered are all situations in which a group of agents has entitlements over a resource that cannot be jointly honored.

Our search for answers begins with the description of several rules that are commonly used in practice or have been discussed in the theoretical literature. Then – and this constitutes the bulk of our work – we formulate properties that one may want rules to satisfy; we compare the rules on the basis of the properties they enjoy; we investigate the existence of rules satisfying various combinations of the properties; and, when rules exist that satisfy a given list of properties, we describe the family they constitute. These properties are formally stated as *axioms*. Finally, we appeal to the conceptual apparatus of modern game theory to construct rules. Both the cooperative branch of the field and its strategic branch are rich in concepts and techniques that proved very helpful in our endeavor.

Only one good is to be allocated here and, for all agents, more is preferred to less. Thus, agents' preferences are the same and they do not appear explicitly in our model. This is an important way in which the class of problems we investigate should be distinguished from other classes most often considered in the theory of economic design. When allocating resources on which agents have equal rights, the issue is typically how best to take account of how their preferences differ. Here, by contrast, agents differ only to the extent that their rights and identities differ. The central question in the discipline of economics is commonly stated as pertaining to the allocation of scarce and valuable resources and, in its standard specification, the reason for scarcity is that preferences are non-satiated. Preferences do not reach satiation here but there are natural upper bounds on consumptions – no agent should be assigned more than their claim – and our focus is on situations where there is not enough to reach these bounds simultaneously. Also, we will not take into account the intensity of the satisfaction that claimants derive from their assignments, as captured by what are usually called "utility functions." In spite of these significant differences, many of the general ideas that underlie properties of rules studied in other contexts are just as pertinent to the understanding of how to adjudicate conflicting claims. They will be fundamental concepts in our evaluation of candidate rules, and they will direct our search for the most desirable ones.

The best-known rule is the "proportional" rule, for which awards are proportional to claims. In fact, proportionality is often taken as the definition of fairness for claims problems. It was already so for Aristotle. But is there any reason to believe the proportional rule superior to the others? Beside Aristotle, an important source of inspiration for the work we present here is ancient literature, such as the Talmud, and a number of medieval authors, in particular Maimonides, where numerical examples are described, and for these examples, recommendations are made that conflict with proportionality. Can these recommendations be rationalized by means of well-behaved rules? The answer is yes, and we will exhibit such rules. Are there grounds for preferring one or the other to the proportional rule and to the other rules that have been proposed more recently? Here, the answer is more complex. We will indeed produce interesting axiomatic underpinnings for a rule that accounts for all of the numbers in the Talmud, and uncover good reasons to promote certain rules encountered in medieval texts, or inspired by these texts, as well as newly defined rules. We will also find that the proportional rule does satisfy many appealing properties, and, in fact, it will frequently emerge out of our axiomatic analysis. So will two rules found in Maimonides. However, a central conclusion to be drawn from our investigation is that, depending upon the viewpoint taken and the intended application of the theory, one or the other of several rules is preferable. On the other hand, certain a priori reasonable rules have rarely come out of axiomatic considerations, and some not at all. This should not be seen as a fatal flaw of these rules, but it diminishes their appeal to a degree.

Real-life claims problems are of course more complicated affairs than our stylized model can adequately represent, but many enlightening lessons can still be learned from its study. Besides, it can be enriched in a number of ways so as to accommodate additional features of resource allocation conflicts that are relevant to their resolution in practice, as we will indicate in various places. A concluding chapter lists even more significant ways in which it can be further generalized.

An important question that we will not address is the extent to which the choice of a particular division rule affects agents' incentives to make commitments that, in the end, one party may be unable to honor. In the context of bankruptcy, these are the incentives to loan and to borrow. Think of a legislature considering reforming bankruptcy laws so as to bring about some goal deemed socially desirable: a higher rate of investment, for example. Enhancing the safety of investing for certain categories of individuals might be achieved by the choice of particular rules, and this legislature would have to take such incentives into account. In many of the other applications, the parameters of the problems to be solved also result from decisions that agents have made in the past. Whatever rule they know would be used at the division stage will in general have had an effect on these earlier choices. In order to handle these kinds of issues, we would need to work with a more general model than the one that is our focus. Risk-taking, effort, and other variables under the control of agents, such as lenders, borrowers, tax payers, government agencies, and others, would have to be explicitly described, stochastic returns to economic activities factored in, and so on. But the theory developed here, which mostly ignores incentives, is a necessary component of the comprehensive treatment – it would have to be formulated in a general-equilibrium and game-theoretic framework – that we envision.[1]

1.2 THE MODEL

Here is the formal model. An amount $E \in \mathbb{R}_+$ of an infinitely divisible resource, the **endowment**, has to be allocated among a **group** $\boldsymbol{N}$ **of agents** having **claims** on it, $\boldsymbol{c_i} \in \mathbb{R}_+$ being the claim of agent $i \in N$, our generic agent. Up to Chapter 10, we take N to be a finite and fixed subset of the set of natural numbers $\mathbb{N}$, usually $\{1, \ldots, n\}$. Using the notation $\mathbb{R}^N_+$ for the cross-product of $|N|$ copies of $\mathbb{R}_+$ indexed by the members of N,[2] the claims vector $\boldsymbol{c} \equiv (c_i)_{i \in N}$ is therefore an element of $\mathbb{R}^N_+$. To complete the model, we add that the endowment is not sufficient to fully honor all claims.

In summary, a claims problem, or simply a **problem**, is a pair $(\boldsymbol{c}, \boldsymbol{E}) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum c_i \geq E$. Let $\mathcal{C}^N$ denote the domain of all problems.[3]

Figures 1.1 and 1.2 illustrate the definition for $N \equiv \{1, 2\}$ and $N \equiv \{1, 2, 3\}$ respectively.

In Chapters 10–12 and in Section 15.3 we extend the model so as to allow the population of claimants to vary and generalize the notation accordingly.

Although the model just described is extremely simple, it is rich enough to be given several interesting and diverse interpretations, and it is mathematically nontrivial, as we will see.

[1] Steps in these various directions are taken by Araujo and Páscoa (2002), Karagözoğlu (2014), and Kıbrıs and Kıbrıs (2013). They are briefly discussed in Section 15.7.

[2] Alternatively, the superscript N may refer to a set pertaining to the agents in N. Which interpretation is intended should be unambiguous from the context.

[3] We allow the equality $\sum c_i = E$ for convenience, although in this boundary case, all claims can in fact be honored.

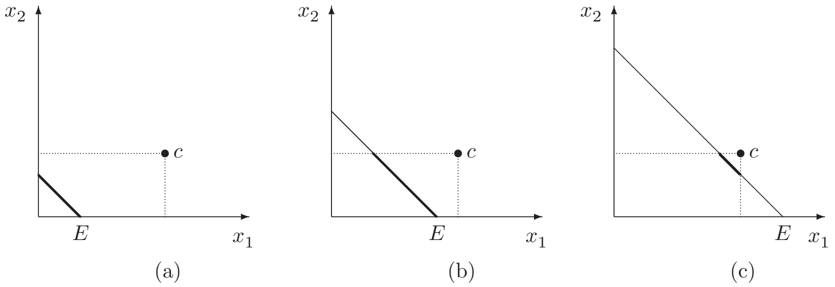

Figure 1.1: **Identifying the awards vectors of a two-claimant problem.** The claimant set is $N \equiv \{1, 2\}$. The claims vector is $c \in \mathbb{R}^N_+$. The endowment is $E \in \mathbb{R}_+$. Axes are indexed by claimants. Along each axis we measure an agent's claim and possible awards to that agent. The objective is to choose a vector satisfying the following conditions: it should be nonnegative and bounded above by c, and it should be on the line of equation $\sum x_i = E$. These are the "awards vectors" of (c, E). In this sequence of panels, c is fixed and E is given three values. In each case, the thick segment represents the set of vectors to choose from. (a) Here, there is no awards vector at which even one claimant is fully compensated. (b) Here, claimant 2 could be fully compensated, if alone; claimant 1 could not. (c) Here, each claimant could be fully compensated, if alone.

One application, already mentioned, is to bankruptcy: there, E is the liquidation value of a bankrupt firm, and each coordinate of c represents the claim held against it by one of its creditors.

A closely related application is to estate division: a man dies and his estate is insufficient to cover the debts he leaves behind. How should it be divided among his creditors? We will sometimes refer to such situations, often discussed in ancient literature, and use then the language of estate division.

The financial decisions faced by the organizer of a scientific meeting whose budget is too limited to fully cover the expenses of all participants, a situation familiar in academia, is another example.

Our next application is to rationing: a group of customers of a firm has placed orders for a good produced by the firm, but the total quantity it can supply turns out to be insufficient to satisfy everyone; orders can only be partially filled. Being able to demonstrate that it has done its best to be even-handed in dealing with the situation might help the firm remain on good terms with all of its customers. So how much should it assign to each of them?

The problem can also occur at the level of nations, when a scarce resource has to be distributed to states or provinces – food, clean water, medical supplies, or shares of the global carbon budget[4] come to mind here – and at the multinational level. For instance, an international agency distributing aid to impoverished countries rarely has enough to cover all of these needs.

[4] Giménez-Gómez, Teixedó-Figueras, Vilella (2016).

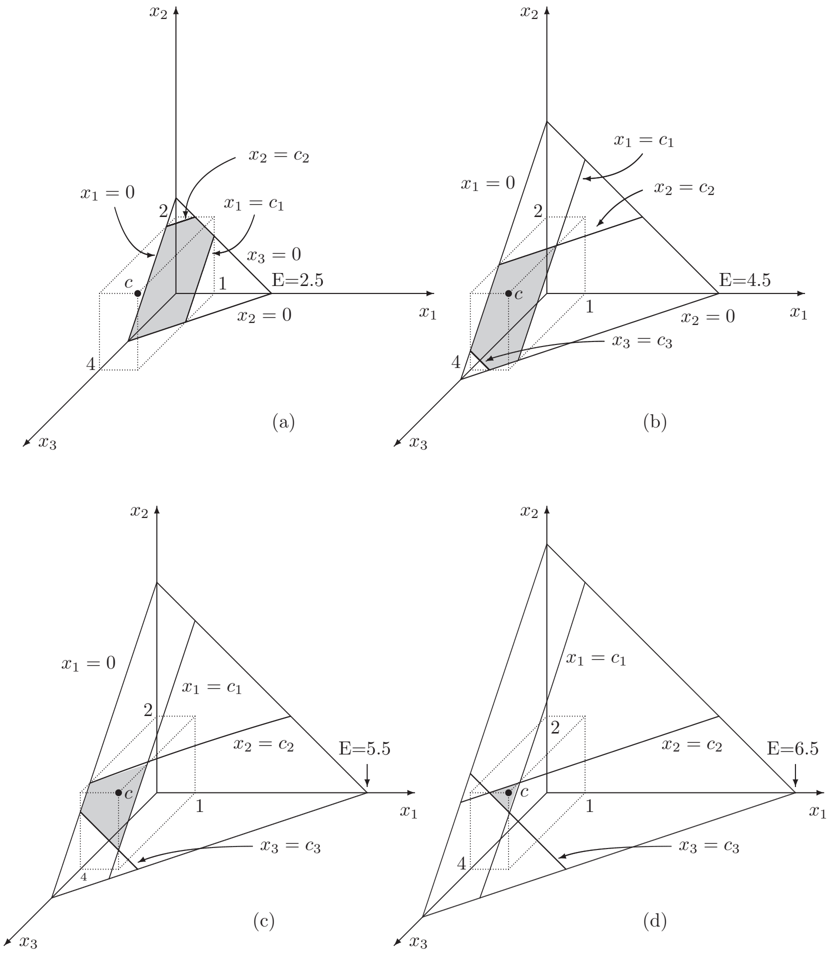

Figure 1.2: **Identifying the awards vectors of a three-claimant problem.** The claimant set is $N \equiv \{1, 2, 3\}$. The claims vector is $c \equiv (1, 2, 4) \in \mathbb{R}^N_+$ and the endowment E is given four values – 2.5 in panel (a), 4.5 in panel (b), 5.5 in panel (c), and 6.5 in panel (d). In each case, the shaded area represents the set of awards vectors of (c, E): they are nonnegative, bounded above by c, and belong to the plane of equation $\sum x_i = E$. Among the vectors $x \in \mathbb{R}^N$ satisfying these conditions, the subset of vectors such that $x_1 = c_1$, say, is indicated by a line in the plane of equation $\sum x_i = E$ labeled "$x_1 = c_1$." Only the constraints that are binding in identifying the set of awards vectors are labeled.

More generally, our model encompasses any situation in which some amount of a resource has to be allocated among a group of agents when that amount is insufficient to satisfy their commensurable claims, needs, or demands.

In most of the applications just listed, these parameters are objective amounts, typically supported by legal documents. Alternatively, they could be given a subjective interpretation, and their values could even be a matter of debate. In the case of needs, for instance, one can certainly imagine experts disagreeing on the aid required by different countries facing medical emergencies. Nevertheless, some reasonable brackets may well be agreed upon, the question being how to select a division that achieves the best compromise among these approximate measures of need. To simplify the analysis we work with precise values of all variables of the model instead of with intervals or distributions.[5]

In our personal and professional lives, misunderstandings sometimes occur as to how much of some resource each of us is entitled to. What should we do? In such situations claims are not supported by legal documents, but it doesn't necessarily mean that they should be ignored. The behavior of agents is often affected by informal agreements and subjective views of the situation they are facing, not just by objective factors. When claims are made in good faith, everyone involved may accept that they should be taken into consideration. In practice, expectations, beliefs, perceptions of fairness, and so on, play an important role in how agents respond to proposed settlements. Thus, the use of a rule whose good behavior can be demonstrated should contribute to dissipate conflicts, and help societies to function more smoothly.

Alternatively, one can simply think of each of the coordinates of the claims vector as a bound on the consumption of the agent to which it pertains, a bound that should not be exceeded for reasons that need not be explicitly specified.

Our model can also be interpreted as a formalization of a simple class of tax assessment problems: there, agents are taxpayers whose incomes are given by the coordinates of c, and who among themselves must cover the cost E of a project. The sum of their incomes is larger than the cost: they can afford the project. The question is how much each taxpayer should contribute.[6] This application differs from the previous one in that what is to be divided is a collective obligation (an agent's welfare decreases when their share of the dividend increases). This difference has no significant mathematical consequences for the theory, but this alternative interpretation of the variables should be kept in mind when evaluating axioms and rules.

Finally, we consider the closely related problem of cost allocation. Now, agents are the users of a public project. The parameter c_i represents the benefit agent i derives from the project, and E is the cost of undertaking it. The sum

5 A model in which the parameters of the problem are intervals is studied by Branzei and Alparslan (2008). See Section 15.7 for a discussion and additional references.

6 Note, however, that in practice the problem of taxation is not generally specified by first stating an amount to be collected, perhaps due to the uncertainty pertaining to the taxpayers' incomes. In most cases, taxation schedules are published first, and the amount collected falls wherever it may, depending upon realized incomes.

of the benefits is larger than the cost, indicating that the project is worth undertaking. How much should each user contribute? This problem has been the subject of a considerable literature, both normative and strategic. An issue that has preoccupied many investigators is that users may misrepresent the benefits they derive from the project, resulting in a distorted decision. The question then is how to elicit the information needed for the correct decision to be made (undertake it if the sum of the valuations exceeds the cost, and not otherwise; achieve a desirable distribution of welfare among the participants).

For convenience, in most of our treatment of the problem, we maintain the interpretation of the model as pertaining to the adjudication of conflicting claims, and we use language that fits that interpretation.

We are tasked with identifying a list of "awards," one for each claimant, whose sum is equal to the endowment. Instead of considering each problem separately, however, we will look for a general method of handling all problems – that is, for a function that associates with each problem a division of the endowment among the claimants. We will require that each claimant receive an amount that is nonnegative and at most as large as their claim. The division is to be thought of as a recommendation for the problem.

Formally, an **awards vector for $(c, E) \in \mathcal{C}^N$** is a vector $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and satisfying the **balance** requirement $\sum x_i = E$.[7] Let $X(c, E)$ be the set of awards vectors for (c, E). A division rule, or simply a **rule**, is a function that associates with each claims problem $(c, E) \in \mathcal{C}^N$ an awards vector for it; that is, a vector in $X(c, E)$. Our generic notation for a rule is the letter S.

We stress that a rule is a **single-valued** mapping, that is, a rule selects a unique awards vector for each problem. This is desirable because it means that the issue of how much to assign to each claimant has been completely resolved. Among the various recommendations that a multi-valued mapping may make for a given problem, on what grounds should one choose? *Single-valuedness* is particularly justified here because, for our model, a great variety of interesting mappings enjoy the property. This fact is worth emphasizing. Indeed, *single-valuedness* is a luxury that one can rarely afford: in most other types of allocation problems, it comes at a high price, excluding many natural mappings or preventing certain appealing properties from being satisfied by any mapping.

The set of awards vectors of each problem is a convex set. Thus, an arbitrary convex combination of rules is a rule.[8] This observation will shed much light on the structure of the space they constitute.

[7] Vector inequalities: $x \geqq y$ allows x and y to be equal; $x \geq y$ does not; $x > y$ means that each coordinate of x is larger than the corresponding coordinate of y.

[8] By the convex combination of two rules S^1 and S^2 with weights $(\lambda^1, \lambda^2) \in \Delta^1$ (where Δ^1 is the unit simplex of $\mathbb{R}^2_+$), we mean the rule that associates with each problem $(c, E) \in \mathcal{C}^N$ the awards vector $\lambda^1 S^1(c, E) + \lambda^2 S^2(c, E)$.

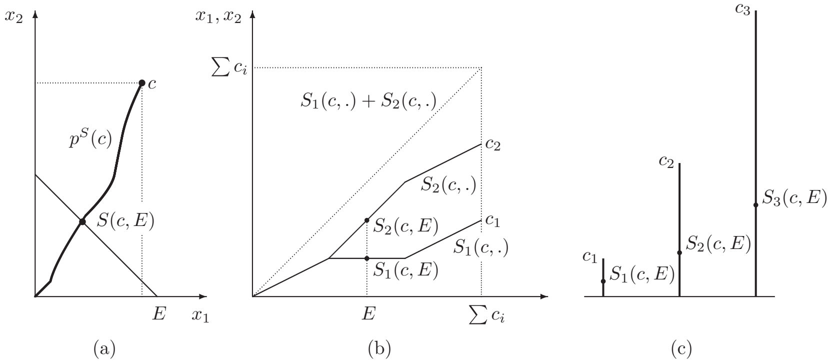

Figure 1.3: **Three ways of depicting a division rule.** In each case, the claims vector is fixed at $c \in \mathbb{R}^N_+$. (a) Here, $N \equiv \{1, 2\}$. We follow the awards vector $S(c, E)$ as E varies from 0 to $\sum c_i$, thereby obtaining the "path of awards" of S for c, $p^S(c)$. Diagrams of this sort are very useful for two claimants and for three claimants. (b) Here, $N \equiv \{1, 2\}$ also. We plot each claimant's award separately as a function of E, measured on the horizontal axis, as E varies from 0 to $\sum c_i$. The sum of the functions is the identity. These "schedules of awards" of S for c can be drawn for any number of claimants. (c) Here, $N \equiv \{1, 2, 3\}$. We represent claims and awards as vertical segments, and given $E \leq \sum c_i$, we indicate by dots the amounts that S awards to the various claimants. This amount is the vertical distance from the horizontal line to the dot. These diagrams can accommodate any number of claimants but only a few values of the endowment.

Figure 1.3 shows three ways of depicting a typical rule, called S. For the important class of Young's rules of Subsection 2.2.2, below, a fourth representation is possible. However, we will wait until then for a formal introduction of the concept and a description of these representations.[9] In each of the panels of Figure 1.3, the claims vector is kept fixed.

1. Graphs of the type represented in panel (a) give, in a Euclidean space of dimension equal to the number of claimants, the path followed by the awards vector chosen by S as the endowment increases from 0 to the sum of the claims. In "awards space," the award to each claimant is measured along the claimant's own axis. We call this path **the path of awards of the rule for the claims vector**. We use the notation $\boldsymbol{p^S(c)}$ for the path of S for c. In our illustrations, we almost always draw paths of awards as continuous, this property being very natural (we formally introduce continuity requirements on rules in Section 3.2). Moreover, as we will see, continuity with respect to the endowment is in fact satisfied by all interesting rules, and we have encountered no

[9] Hendrickx, Borm, van Elk, and Quant (2005) propose to represent a rule by specifying, for each $c \in \mathbb{R}^N_+$ and each $i \in N$, a function defined on $[0, \sum c_i]$ whose integral, for each E in that interval, gives claimant i's award.

situation where it has a cost in terms of other properties. However, paths could in principle be discontinuous.

Paths of awards are most suggestive of the behavior of rules. For example, we will discover that a rule can very usefully be evaluated by assessing, for each claims vector, how close to the 45° line – or to the ray emanating from the origin and passing through the claims vector – its path lies, whether it exhibits any concavity or convexity property, whether it is smooth or has kinks, and so on. Such qualitative features are revealed by simple visual inspection. Depicting rules by means of their paths of awards is also very useful for proofs. Paths are harder to draw and to visualize for three claimants, but we will still find them very useful. Of course, this representation cannot accommodate any larger number of claimants, but most of the difficulties in developing our theory already occur for two or three claimants.

2. Graphs of the type illustrated in panel (b) can handle an arbitrary number of claimants: for each claims vector, we simply plot the award chosen by the rule for each claimant as a function of the endowment. The domain of definition of this function is the interval from 0 to the sum of the claims. The sum of the functions is the identity function. We call these plots **schedules of awards of the rule for the claims vector**. The functions could be discontinuous too, but once again, in our illustrations, we almost always draw them continuous.

3. On a graph of the type illustrated in panel (c), where claims and awards are represented as vertical segments, only a few of the awards vectors chosen by a rule can be shown without clutter (for no more than three or four choices of the endowment), but such graphs are nevertheless very convenient for certain proofs. In particular, they too can accommodate an arbitrary number of claimants. (On rare occasions, we will prefer to represent claims as horizontal segments as opposed to vertical segments.)

1.3 TWO PUZZLES IN THE TALMUD

To someone not familiar with economic design, the need to go beyond whatever is generally done in practice, in our case proportionality, is not always obvious. However, we hope to convince readers of the great benefit of making a clean slate of any preconceived notions about which rules are better. Besides, proportionality is not as universal as one may think. Although it has been advocated since Aristotle, we describe in this section several problems discussed in the Talmud[10] for which the Talmud does not recommend proportional division. Only a few numerical examples are specified there, but several other examples appear in ancient literature, which we will describe in due time and for which outcomes other than the proportional outcomes were suggested. We expect that these intriguing examples will whet our readers' appetite as much as they have whetted the appetite of many of the researchers who have contributed to the subject.

[10] The Talmud is the collection of writings that constitute the basis for Jewish Law.

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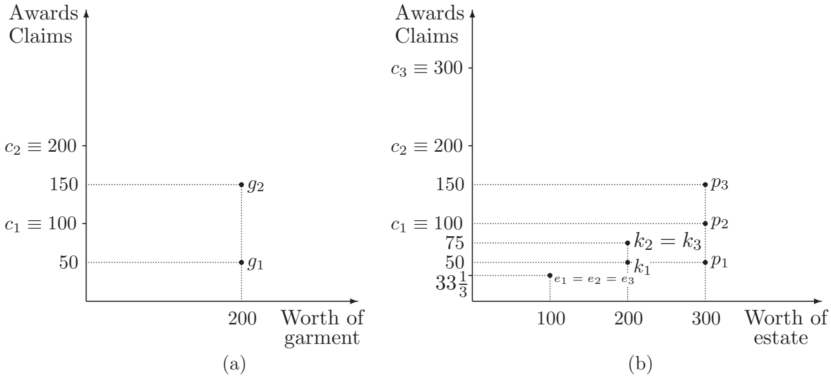

Figure 1.4: **Two puzzles in the Talmud**. The numbers in the Talmud that have to be explained pertain to the problems represented here. (a) In the "contested garment" problem there are two claimants, with claims 100 and 200, over a garment worth 200. The Talmud recommends the division $g \equiv (50, 150)$. (b) In the "marriage contract" problems, there are three claimants with claims 100, 200, and 300. If the estate is worth 100, the Talmud recommends $e \equiv (33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$; if worth 200, it recommends $k \equiv (50, 75, 75)$; and if worth 300, it recommends $p \equiv (50, 100, 150)$. Can one make sense of these choices?

The **contested garment problem** (Figure 1.4a) has to do with two men disagreeing over the ownership of a garment and making incompatible claims on it. How should the garment – rather its worth – be divided between them? Here is the description of the problem and the recommendation made for it:[11]

> Two hold a garment... If one of them says, "It is all mine," and the other says "Half of it is mine," ... the former receives three quarters and the latter receives one quarter.

To fix the ideas, let us assign a worth of 200 to the garment. Then, one man claims 100 and the other claims 200. Thus, the suggestion made in the Talmud is the division (50, 150).[12]

[11] See Baba Metzia, Babylonian Talmud I. All references to the relevant passages of the Talmud and of the secondary literature are taken from O'Neill (1982), Aumann and Maschler (1985), and Dagan (1996). For additional citations, see Callen (1987) and Aumann (2010).

[12] A variant of this numerical example also appears in a Tosefta to Baba Metzia, in which the smaller claim is one-third of the garment, the larger claim still being the entire garment. The recommendation there is that the smaller claimant gets one-sixth of the garment and the other the remainder. Another example in which two principles of liability conflict is in Baba Kamma 53a. First, the owner of a wild ox is responsible for half of the damages the ox may cause. Second, someone having dug an open pit on public property is liable for all of the damages it may cause. The example involves a wild ox causing an animal to fall in an open pit. The numerical values attached to the example are as in the contested garment problem and the recommendations made for it in the Talmud are the same as for the contested garment problem. References and a discussion are in Aumann (2010).