1

Motivation and Background

An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.

— Robert P. Langlands[1]

This first chapter serves as a *tour d'horizon* of the topics covered in this book. Its purpose is to introduce and survey the key concepts relevant for the study of automorphic forms and automorphic representations in the adelic language (Part One of the book), their application in string theory (Part Two) and how they relate to other questions of current research interest in mathematics and physics (Part Three).

1.1 Automorphic Forms and Eisenstein Series

Automorphic forms are complex functions $f(g)$ on a real Lie group G that

(1) are invariant under the action of a discrete subgroup $\Gamma \subset G$: $f(\gamma \cdot g) = f(g)$ for all $\gamma \in \Gamma$,
(2) satisfy eigenvalue differential equations under the action of the ring of G-invariant differential operators, and
(3) have well-behaved *growth conditions*.

[1] Mostow, G. D. and Caldi, D. G. (eds) 1989. *Representation Theory: Its Rise and Its Role in Number Theory (Proceedings of the Gibbs Symposium)*. American Mathematical Society, Providence, RI.

A more explicit and refined form of these conditions will be given in Chapter 4 when we properly define automorphic forms; here we content ourselves with a qualitative description based on examples. We will mainly be interested in automorphic forms $f(g)$ that are invariant under the action of the *maximal compact subgroup* K of G when acting from the right: $f(gk) = f(g)$ for all $k \in K$; such forms are called *K-spherical*. The automorphic forms are then functions on the coset space G/K.

The prime example of an automorphic form is obtained when considering $G = SL(2,\mathbb{R})$ and $\Gamma = SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$. The maximal compact subgroup is $K = SO(2,\mathbb{R})$ and the coset space G/K is a real two-dimensional constant negative curvature space isomorphic to the *Poincaré upper half-plane*,

$$\mathbb{H} = \{z = x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}. \tag{1.1}$$

A function satisfying the three criteria above is then given by the non-holomorphic function with $s \in \mathbb{C}$,

$$f_s(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}}. \tag{1.2}$$

The sum converges absolutely for $\mathrm{Re}(s) > 1$. The action of an element $\gamma \in SL(2,\mathbb{Z})$ on $z \in \mathbb{H}$ is given by the standard fractional linear form (see Section 4.1)

$$\gamma \cdot z = \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{1.3}$$

Property (1) is then verified by noting that the integral lattice $(c,d) \in \mathbb{Z}^2$ is preserved by the action of $SL(2,\mathbb{Z})$ and acting with $\gamma \in SL(2,\mathbb{Z})$ on (1.2) merely reorders the terms in the absolutely convergent sum. Property (2) in this case reduces to a single equation since there is only a single primitive $SL(2,\mathbb{R})$-invariant differential operator on $SL(2,\mathbb{R})/SO(2,\mathbb{R})$. This operator is given by

$$\Delta = y^2 \left(\partial_x^2 + \partial_y^2\right) \tag{1.4}$$

and corresponds to the (scalar) *Laplace–Beltrami operator* on the upper half-plane $\mathbb{H}$. In group theoretical terms it is the quadratic Casimir operator. Acting with it on the function (1.2), one finds

$$\Delta f_s(z) = s(s-1) f_s(z) \tag{1.5}$$

and hence $f_s(z)$ is an eigenfunction of Δ (and therefore of the full ring of differential operators generated by Δ). Condition (3) relating to the growth of the function here corresponds to the behaviour of $f_s(z)$ near the boundary of the upper half-plane, more particularly near the so-called *cusp at infinity* when $y \to \infty$.[2] The growth condition requires $f_s(z)$ to grow at most as a power law as $y \to \infty$. To verify this point it is easiest to consider the *Fourier expansion* of $f_s(z)$. This requires a bit more machinery and also paves the way to the general theory. We will introduce it heuristically in Section 1.3 below and in detail in Chapter 6.

The form of the function $f_s(z)$ is very specific to the action of $SL(2,\mathbb{Z})$ on the upper half-plane $\mathbb{H}$. To prepare the ground for the more general theory of automorphic forms for higher-rank Lie groups we shall now rewrite (1.2) in a more suggestive way. In fact, $f_s(z)$ is (almost) an example of a (non-holomorphic) *Eisenstein series* on $G = SL(2,\mathbb{R})$. To see this, we first extract the greatest common divisor of the coordinates of the lattice point $(c,d) \in \mathbb{Z}^2$:

$$f_s(z) = \left(\sum_{k>0} k^{-2s}\right) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}} = \zeta(2s) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}, \tag{1.6}$$

where we have evaluated the sum over the common divisor k using the *Riemann zeta function* [541]

$$\zeta(s) = \sum_{n>0} n^{-s}. \tag{1.7}$$

Referring back to (1.3), we can rewrite the summand using an element γ of the group $SL(2,\mathbb{Z})$:

$$\frac{y^s}{|cz+d|^{2s}} = [\operatorname{Im}(\gamma \cdot z)]^s \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.8}$$

For interpreting all summands in (1.6) in this way, two things have to occur: (i) for any coprime pair (c,d) such a matrix $\gamma \in SL(2,\mathbb{Z})$ must exist, and (ii) if several matrices exist we must form equivalence classes such that the sum over coprime pairs (c,d) corresponds exactly to the sum over equivalence classes. For (i), we note that the condition that c and d be coprime is necessary since it would otherwise be impossible to satisfy the determinant condition $ad - bc = 1$ over $\mathbb{Z}$. At the same time, co-primality is sufficient to guarantee existence of integers a_0 and b_0 that complete c and d to a matrix $\gamma \in SL(2,\mathbb{Z})$. In fact, there is a one-parameter family of solutions for γ that can be written as

$$\begin{pmatrix} a_0 + mc & b_0 + md \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c & d \end{pmatrix} \tag{1.9}$$

[2] For $\Gamma = SL(2,\mathbb{Z})$ this is the only cusp up to equivalence. With this, one means that any fundamental domain of the action of Γ on $\mathbb{H}$ only touches the boundary of the upper half-plane at a single point. See Section 4.1 for illustrations and [439, 417, 73] for more details on discrete subgroups of $SL(2,\mathbb{R})$.

for any integer $m \in \mathbb{Z}$. (That these are all solutions to the determinant condition over $\mathbb{Z}$ is an elementary lemma of number theory, sometimes called *Bézout's lemma* [380].) The form (1.9) tells us also how to resolve point (*ii*): we identify matrices that are obtained from each other by left-multiplication by a matrix belonging to the *Borel subgroup*

$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} \middle| m \in \mathbb{Z} \right\} \subset SL(2,\mathbb{Z}). \tag{1.10}$$

The interpretation of this group is that it is the stabiliser of the x-axis (or the cusp at infinity).

Summarising the steps we have performed, we find that we can write the function (1.2) as

$$f_s(z) = 2\zeta(2s) \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \left[\operatorname{Im}(\gamma \cdot z)\right]^s . \tag{1.11}$$

Since we included the matrix $-\mathbb{1}$ in the definition of $B(\mathbb{Z})$, an extra factor of 2 arises in this formula.

Dropping the normalising zeta factor, we obtain the function

$$E(\chi_s, z) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi_s(\gamma \cdot z), \tag{1.12}$$

where we have also introduced the notation $\chi_s(z) = [\operatorname{Im}(z)]^s = y^s$. The reason for this notation is that χ_s is actually induced from a *character* on the real Borel subgroup. We will explain this in detail below in Chapter 4. Note that this way of writing the automorphic form makes the invariance under $SL(2,\mathbb{Z})$ completely manifest because it is a *sum over images*.

The form (1.12) is what we will call an *Eisenstein series* on $SL(2,\mathbb{R})$ and it is this form that generalises straightforwardly to Lie groups $G(\mathbb{R})$ other than $SL(2,\mathbb{R})$ (whereas the form with the sum over a lattice does not, as we discuss in more detail in Section 15.3). In complete analogy with (1.12) we define the *(minimal parabolic) Eisenstein series* on $G(\mathbb{R})$ invariant under the discrete group $G(\mathbb{Z})$ by[3]

$$E(\chi, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi(\gamma g), \tag{1.13}$$

where χ is (induced from) a character on the Borel subgroup $B(\mathbb{R})$ and $g \in G(\mathbb{R})$. Eisenstein series are prototypical automorphic forms and the protagonists of the story we shall develop.

[3] For most of this book, we shall take $G(\mathbb{Z})$ as the discrete Chevalley subgroup of an algebraically simply-connected group split over the real numbers; see Section 3.1.4 below for more details.

1.2 Why Eisenstein Series and Automorphic Forms?

Before delving into the further analysis of Eisenstein series and automorphic forms, let us briefly step back and provide some motivation for their study.

1.2.1 A Mathematician's Possible Answer

Automorphic forms are of great importance in many fields of mathematics such as number theory, representation theory and algebraic geometry. The various ways in which automorphic forms enter these seemingly disparate fields are connected by a web of conjectures collectively referred to as the Langlands program [444, 446, 259, 412, 413, 219] that we will discuss more in Chapter 16.

Much of the arithmetic information is contained in the Fourier coefficients of automorphic forms. The standard examples correspond to modular forms on $G(\mathbb{R}) = SL(2,\mathbb{R})$, where these coefficients yield eigenvalues of Hecke operators (covered in Chapter 11) and the counting of points on elliptic curves; see Section 16.4.

For arbitrary Lie groups $G(\mathbb{R})$ one considers the Hilbert space $L^2(\Gamma\backslash G(\mathbb{R}))$ of square-integrable functions that are invariant under a left-action by a discrete subgroup $\Gamma \subset G(\mathbb{R})$. This space carries a natural action of $g \in G(\mathbb{R})$, called the *right-regular action*, through

$$[\pi(g)f](x) = f(xg), \tag{1.14}$$

where $f \in L^2(\Gamma\backslash G(\mathbb{R}))$, $g, x \in G(\mathbb{R})$ and $\pi \colon G(\mathbb{R}) \to \mathrm{Aut}\big(L^2(\Gamma\backslash G(\mathbb{R}))\big)$ is the right-regular representation map. Since the functions are square-integrable the representation is unitary. This representation-theoretic viewpoint on automorphic forms was first proposed by Gelfand, Graev and Piatetski-Shapiro [267] and later developed considerably by Jacquet and Langlands [376]. This perspective provides the key to generalising the classical theory of modular forms on the complex upper half-plane to higher-rank Lie groups.

It is an immediate, important and difficult question as to what the decomposition of the space $L^2(\Gamma\backslash G(\mathbb{R}))$ into irreducible representations of $G(\mathbb{R})$ looks like. The irreducible constituents in this decomposition are called *automorphic representations*. This spectral problem was tackled and solved by Langlands [447]. The Eisenstein series (and their analytic continuations) form an integral part in the resolution although they themselves are not square-integrable.[4] We shall discuss aspects of this in Chapter 5.

[4] A passing physicist might note that this is very similar to using non-normalisable plane waves as a 'basis' for wave functions in quantum mechanics. Indeed the piece $\chi(\gamma g)$ in (1.13) is exactly like a plane wave; the γ-sum is there to make it invariant under the discrete group by the method of images so that $E(\chi, g)$ are the simplest Γ-invariant plane waves. The decomposition

1.2.2 A Physicist's Possible Answer

Many problems in quantum mechanics are characterised by discrete symmetries. At the heart of many of them lies *Dirac quantisation*, where charges (e.g., electric or magnetic) of physical states are restricted to lie in certain lattices rather than in continuous spaces. The discrete symmetries preserving the lattice are often called *dualities* and can give very interesting different angles on a physical problem. This happens in particular in string theory (see Part Two), where such dualities can mix perturbative and non-perturbative effects.

For the discrete symmetry to be a true symmetry of a physical theory, all observable quantities must be given by functions that are invariant under the discrete symmetry, corresponding to property (1) discussed at the beginning of Section 1.1. Similarly, the dynamics or other symmetries of the theory impose differential equations on the observables, corresponding to property (2), and the growth condition (3) is typically associated with having well-defined perturbative regimes of the theory. The main example we have in mind here comes from string theory and the construction of scattering amplitudes of type II strings in various maximally supersymmetric backgrounds, as is discussed in Part Two of this book. However, the logic is not necessarily restricted to this; see also [592, 515, 65, 33] for some other uses of automorphic forms in physics.

For these reasons one is often led naturally to the study of automorphic forms in physical systems with discrete symmetries. Via this route one also encounters a spectral problem similar to the one posed by mathematicians when one needs to determine to which automorphic representation a given physical observable belongs. Again, the Eisenstein series and their properties are important building blocks of such spaces and it is important to understand them well. Furthermore, in a number of examples from string theory it was actually possible to show that the observable is given by an Eisenstein series itself [308, 311, 321, 407, 503, 318, 313, 520, 315].[5]

1.3 Analysing Automorphic Forms and Adelisation

We now return to the study of Eisenstein series defined by (1.13) and their properties, starting again with the very explicit example (1.2) for $SL(2,\mathbb{R})$.

of an automorphic function ('wave-packet') in this basis (extended by the discrete spectrum of cusp forms and residues) is the content of various trace formulas discussed in Section 5.5.

5 That Eisenstein series are mostly not square-integrable is no problem in these cases since the object computed (part of a scattering amplitude) is not a wavefunction and not required to be normalisable.

1.3.1 Fourier Expansion of the $SL(2,\mathbb{R})$ Series

The discrete Borel subgroup $B(\mathbb{Z})$ of (1.10) acts on the variable $z = x + iy$ through translations by

$$\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} \cdot z = z \pm m \quad \text{for } m \in \mathbb{Z} \tag{1.15}$$

and therefore any $SL(2,\mathbb{Z})$-invariant function is periodic in the x direction with period equal to 1 corresponding to the smallest non-trivial $m = 1$. This means that we can Fourier expand it in modes $e^{2\pi i n x}$. Applying this to (1.12) leads to

$$E(\chi_s, z) = \underbrace{C(y)}_{\substack{\text{constant term} \\ \text{zero mode}}} + \underbrace{\sum_{n\neq 0} a_n(y) e^{2\pi i n x}}_{\text{non-zero mode}} . \tag{1.16}$$

As we indicated, it is natural to divide the *Fourier expansion* into two parts depending on whether one deals with the zero Fourier mode (a.k.a. *constant term*) or with a non-zero Fourier mode. Since the Fourier expansion was only in the x-direction, the Fourier coefficients still depend on the second variable y.[6]

Determining the explicit form of the Fourier coefficients is one of the key problems in the study of Eisenstein series. In the example of $SL(2,\mathbb{R})$, this can for instance be done by making recourse to the formulation in terms of a lattice sum that was given in (1.2) and using the technique of *Poisson resummation*. The calculation is reviewed in Appendix A and leads to the following explicit expression:

$$\begin{aligned} E(\chi_s, z) = {} & y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} \\ & + \frac{2y^{1/2}}{\xi(2s)} \sum_{m\neq 0} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}, \end{aligned} \tag{1.17}$$

where

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{1.18}$$

is the *completion of the Riemann zeta function* (1.7) with the standard Γ-function, $K_s(y)$ is the *modified Bessel function of the second kind* (which decreases exponentially for $y \to \infty$ in accordance with the growth condition) and

$$\sigma_{1-2s}(m) = \sum_{d|m} d^{1-2s} \tag{1.19}$$

[6] If one dealt with an automorphic form holomorphic in z (called modular form in Chapter 4 below) this would not be true since the holomorphicity condition links the x and y dependence. The Fourier coefficients in an expansion in $q = e^{2\pi i(x+iy)} = e^{2\pi i z}$ would be pure numbers. This is the origin of the name *constant term* for the zero mode in (1.16).

is called a *divisor sum* (or the *instanton measure* in physics; see Section 13.6.3) and given by a sum over the positive divisors of $m \neq 0$.

As is evident from (1.17), the explicit form of the Fourier expansion can appear quite complicated and involves special functions as well as number-theoretic objects. For the case of more general groups $G(\mathbb{R})$, the method of Poisson resummation is not necessarily available as there is not always a form of the Eisenstein series as a lattice sum. *It is therefore desirable to develop alternative techniques for obtaining (parts of) the Fourier expansion under more general assumptions.*[7] This is achieved by lifting the Eisenstein series into an adelic context which we now sketch and explain in more detail in Section 4.4.

1.3.2 Adelisation of Eisenstein Series

A standard elementary technique in number theory for analysing equations over $\mathbb{Z}$ is by analysing them instead as congruences for every prime (and its powers) separately (sometimes known as the *Hasse principle* or the *local-global principle* based on the *Chinese remainder theorem*) [23, 500]. One way of writing all the terms together is to use the *ring of adeles* $\mathbb{A}$. The adeles can formally be thought of as an infinite tuple

$$a = (a_\infty; a_2, a_3, a_5, a_7, \ldots) \in \mathbb{A} = \mathbb{R} \times \prod_{p<\infty}{}' \mathbb{Q}_p, \tag{1.20}$$

where $\mathbb{Q}_p$ denotes the *p-adic numbers*. The p-adic numbers are a completion of the rational number $\mathbb{Q}$ that is inequivalent to the standard one (leading to $\mathbb{R}$) and that is parametrised by a prime number p and defined properly in Section 2.1. The product is over all prime numbers and the prime on the product symbol indicates that the entries a_p in the tuple are restricted in a certain way (see (2.59) below for the exact statement). The real numbers $\mathbb{R}$ can be written as $\mathbb{Q}_\infty$ in this context and interpreted as the completion of $\mathbb{Q}$ at the 'prime' $p = \infty$. *Very* crudely, an adele can be thought of as summarising the information of an object modulo all primes.

Strong approximation is a similar method that lifts a general automorphic form from being defined on the space $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$ to the space $G(\mathbb{Q})\backslash G(\mathbb{A})/K(\mathbb{A})$ so that $G(\mathbb{Q})$ plays the rôle of the discrete subgroup that was played by $G(\mathbb{Z})$ before. However, $G(\mathbb{Q})$ is a nicer group than $G(\mathbb{Z})$ since $\mathbb{Q}$ is a field whereas $\mathbb{Z}$ is only a ring. This facilitates the analysis and allows the application of many theorems for algebraic groups.

[7] Additional care has to be taken for the Fourier expansion for general $G(\mathbb{R})$ also because the translation group $B(\mathbb{Z})$ is in general not abelian. One can still define (abelian) Fourier coefficients, as we will see; however, they fail to capture the full Eisenstein series. There are also non-abelian parts to the Fourier expansion, as we discuss in Chapter 6.

A consequence of using strong approximation and adeles is that the result of many calculations factorises according to (1.20) and one can do the calculation for all primes and $p = \infty$ separately. Indeed, the explicit form (1.17) for the Fourier expansion of the $SL(2,\mathbb{R})$ Eisenstein series already secretly had this form. This can be seen for example in the constant term, since

$$\frac{\xi(2s-1)}{\xi(2s)} = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\prod_{p<\infty}\frac{1-p^{-2s}}{1-p^{1-2s}}, \tag{1.21}$$

where we have used the definition of the completed zeta function from (1.18) and the *Euler product formula* for the Riemann zeta function [541]

$$\zeta(s) = \sum_{n>0} n^{-s} = \prod_{p<\infty}\frac{1}{1-p^{-s}}. \tag{1.22}$$

In (1.21), we clearly recognise a factorised form that is very similar to (1.20). That this is not an accident will be demonstrated in Section 7.2 for $SL(2,\mathbb{R})$. The factorisation of the (completed) Riemann zeta function itself can also be understood in this way; see Section 2.7. For the other Fourier modes in (1.17) we get a similar factorisation with the modified Bessel function belonging to the $p = \infty$ factor and

$$\sigma_{1-2s}(m) = \prod_{p<\infty}\gamma_p(m)\frac{1-p^{-(2s-1)}\,|m|_p^{2s-1}}{1-p^{-(2s-1)}}, \tag{1.23}$$

where $|m|_p$ is the p-adic norm of m defined in Section 2.1 and $\gamma_p(m)$ selects all factors with $|m|_p \leq 1$ as shown in Section 2.4. The complete derivation for the non-constant terms can be found in Section 7.3 for the $SL(2,\mathbb{R})$ Eisenstein series.

The adelic methods are so powerful that one can obtain a closed, simple and group-theoretic formula for the constant term of Eisenstein series on any (split real) Lie group $G(\mathbb{R})$. This formula, known as the *Langlands constant term formula*, will be the topic of Chapter 8.

For the (abelian) Fourier coefficients, the adelic methods also help to obtain fairly general results, in particular for the part that involves the finite primes $p < \infty$. For the contribution coming from the $\mathbb{R}$ in (1.20) the results are not quite as general; already for $SL(2,\mathbb{R})$ this is what gives the modified Bessel function. We discuss the computation of Fourier coefficients in Chapter 9.

1.4 Reader's Guide and Main Theorems

The following is a brief outline of the contents and results of this book that is divided into three parts. Part One deals with automorphic forms and automorphic representations, Part Two addresses their applications in string theory and Part Three discusses advanced topics of current research in mathematics and physics.

As a general rule, we have placed a lot of emphasis on examples and explicit calculations where possible to illustrate the key concepts. The main theorems are proved in full detail and many advanced ideas are illustrated with the main steps, and references are given for additional information. Topics that are somewhat supplementary to the main thread of the exposition are contained in sections marked with an asterisk, and they are not essential to follow the development of the main theory.

We also would like to give some disclaimers: unless otherwise specified, all groups $G(\mathbb{R})$ that will be considered here are associated with split real forms and we also restrict to algebraically simply-connected groups. Except for certain sections in Part Three, the base field for the ring of adeles will be the rational numbers $\mathbb{Q}$. Often we will perform formal manipulations of infinite sums and integrals without paying attention to whether the expressions are (absolutely) convergent or not. The expressions typically depend on a set of parameters and for some range of parameters convergence can be established, and we assume them to be in this range. In many cases, the results can be extended by analytic continuation.

Contents of Part One: Automorphic Representations

Chapters 2 and 3 are introductory and provide some background material and set the notation for the rest of the book. More precisely, Chapter 2 introduces the basic machinery of p-adic and adelic analysis which will be crucial for almost everything we do later. The main thrust of the chapter is provided by the numerous examples of computing p-adic integrals that will be used extensively in proving Langlands' constant term formula, and computing Fourier coefficients of Einstein series. In Chapter 3, we introduce some basic features of Lie algebras and Lie groups that will be used throughout the book. We first discuss Lie groups and Lie algebras over $\mathbb{R}$, their discrete subgroups and then move on to algebraic groups over $\mathbb{Q}_p$ as well as adelic groups.

The proper discussion of automorphic forms and automorphic representations begins after these preliminaries, and we now summarise the structure of the remainder of Part One in a little more detail, with emphasis on the central results in each chapter.

- In Chapters 4 and 5, we introduce the general theory of automorphic forms and automorphic representations. We start out gently by discussing how to pass from modular forms on the upper half-plane to automorphic forms on the adelic group $SL(2, \mathbb{A})$. We then move on to the general case of arbitrary Lie