Classical Field Theory

Classical field theory, which concerns the generation and interaction of fields, is a logical precursor to quantum field theory and can be used to describe phenomena such as gravity and electromagnetism.

Written for advanced undergraduates, and appropriate for graduate-level classes, this book provides a comprehensive introduction to field theories, with a focus on their relativistic structural elements. Such structural notions enable a deeper understanding of Maxwell's equations, which lie at the heart of electromagnetism, and can also be applied to modern variants such as Chern-Simons and Born-Infeld electricity and magnetism.

The structure of field theories and their physical predictions are illustrated with compelling examples, making this book perfect as a text in a dedicated field theory course, for self-study, or as a reference for those interested in classical field theory, advanced electromagnetism, or general relativity. Demonstrating a modern approach to model building, this text is also ideal for students of theoretical physics.

Joel Franklin is a professor in the Physics Department of Reed College. His work focuses on mathematical and computational methods with applications to classical mechanics, quantum mechanics, electrodynamics, general relativity, and modifications of general relativity. He is author of two previous titles: *Advanced Mechanics and General Relativity* and *Computational Methods for Physics*.

Classical Field Theory

JOEL FRANKLIN
Reed College, Oregon

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107189614
DOI: 10.1017/9781316995419

First published 2017

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalog record for this publication is available from the British Library

ISBN 978-1-107-18961-4 Hardback

For Lancaster, Lewis, and Oliver

Contents

Preface

This is a book on classical field theory, with a focus on the most studied one: electricity and magnetism (E&M). It was developed to fill a gap in the current undergraduate physics curriculum – most departments teach classical mechanics and then quantum mechanics, the idea being that one informs the other and logically precedes it. The same is true for the pair "classical field theory" and "quantum field theory," except that there are almost no dedicated classical field theory classes. Instead, the subject is reviewed briefly at the start of a quantum field theory course. There are a variety of reasons why this is so, most notably because quantum field theory is enormously successful and, as a language, can be used to describe three of the four forces of nature. The only classical field theory (of the four forces of nature) that requires the machinery developed in this book is general relativity, which is not typically taught at the undergraduate level. Other applications include fluid mechanics (also generally absent from the undergraduate course catalogue) and "continuum mechanics" applications, but these tend to be meant primarily for engineers.

Yet classical field theory provides a good way to think about modern physical model building, in a time where such models are relevant. In this book, we take the "bottom up" view of physics, that there are certain rules for constructing physical theories. Knowing what those rules are and what happens to a physical model when you break or modify them is important in developing physical models beyond the ones that currently exist. One of the main points of the book is that if you ask for a "natural" vector field theory, one that is linear (so superposition holds) and is already "relativistic," you get Maxwell's E&M almost uniquely. This idea is echoed in other areas of physics, notably in gravity, where if you similarly start with a second rank symmetric field that is linear and relativistic, and further require the universal coupling that is the hallmark of gravity, you get general relativity (almost uniquely). So for these two prime examples of classical field theories, the model building paradigm works beautifully, and you would naturally develop this pair even absent any sort of experimental observation (as indeed was done in the case of general relativity, and even Maxwell's correction to Faraday's law represents a similar brute force theoretical approach). But we should also be able to go beyond E&M and gravity. Using the same guidelines, we can develop a theory of E&M in which the photon has mass, for example, and probe the physics implied by that change.

Lagrangians and actions are a structure-revealing way to explore a physical theory, but they do not lend themselves to specific solutions. That's why E&M is the primary theory discussed in this book. By the time a student encounters the material presented here, they will have seen numerous examples of solutions to the Maxwell field equations, so that they know what statements like $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, $\nabla \times \mathbf{B} = 0$, mean physically, having introduced various types of charge sources and solved the field equations. What is less clear from Maxwell's equations are notions of gauge freedom and relativistic covariance. Moving the familiar **E** and **B** (or perhaps more appropriately, V and **A**) into a revealing scalar structure, like the E&M action, allows for a discussion beyond the specific solutions that are studied in most introductory E&M courses. As an example, one can easily develop the notion of a conserved stress tensor from the E&M action. Doing that without the action is much harder and less clear (in terms of conservation laws and symmetries).

I have used this text, or pieces of it, as the basis for a second, advanced semester of E&M, complementing a semester of E&M at the level of reference [19]. In such a course, the physical focus is on the acceleration fields that are necessary to describe radiation.[1] Those fields come directly from the Green's function for the wave equation in four-dimensional space-time, so a discussion of Green's functions is reasonable and also represents the beginning and end of the integral solution for field theories that are linear. The goal, in general, is to use advanced elements of E&M to motivate techniques that are useful for all field theories. But it is also possible to teach a dedicated classical field theory course from the book, without over-dependence on E&M as the primary example. There are plenty of additional physical ideas present in the book, including the action and conserved quantities for both Schrödinger's equation and the Klein–Gordon equation, in addition to the latter's interpretation as the scalar relativistic analogue to Schrödinger's equation.

The text is organized into four chapters and three appendixes. Chapter 1 is a relatively standard review of special relativity, with some in-depth discussion of transformation and invariants and a focus on the modified dynamics that comes from using Newton's second law with relativistic momentum rather than the pre-relativistic momentum. In Chapter 2, the focus is on Green's functions, with the main examples being first static problems in electricity, and then the full wave equation of E&M. The role of the Green's function as an integral building block is the focus, and the new radiation fields the physical beneficiary. Chapter 3 reviews Lagrangian mechanics and then introduces the notion of a field Lagrangian and action whose minimization yields familiar field equations (just as the extremization of classical actions yields familiar equations of motion, i.e., the Newtonian ones). The free particle field equations (and associated actions) for scalars, vectors, and tensors are developed, and then the free fields are coupled to both field sources and particle sources. One of the advantages of the scalar action is the automatic conservation of a stress tensor, a good example of the utility of Noether's theorem. The end of the chapter has a discussion of physical model building and introduces Born–Infeld E&M and Chern–Simons E&M. Finally, Chapter 4 is on gravity, another classical field theory. We establish that Newtonian gravity is insufficient (because it is not relativistically covariant) and explore the implications of universal coupling on gravity as a field theory (basically requiring gravity to be represented by a second rank symmetric tensor field to couple to the full stress tensor, with nonlinear field equations, to couple to itself).

[1] Those fields are sufficiently complicated as to be avoided in most undergraduate courses, except in passing. They are difficult for a number of reasons. First, structurally, since they bear little resemblance to the familiar Coulomb field that starts off all E&M investigation. Second, the acceleration fields are analytically intractable except for trivial cases. Even the first step in their evaluation, calculating the retarded time at a field point, requires numerical solution in general.

The appendixes are side notes, and fill in the details for some techniques that are useful for field theories (both classical and quantum). There is an appendix on mathematical methods (particularly complex contour integration, which is good for evaluating Green's functions) and one on numerical methods (that can be used to solve the actual field equations of E&M in the general setting, for example). Finally, there is a short essay that makes up the third appendix and is meant to demonstrate how you can take a compact action and develop from it all the physics that you know from E&M. When I teach the class, I ask the students to perform a similar analysis for a modified action (typically Proca, but one could use any of a number of interesting current ones).

There exist many excellent texts on classical field theory, classics such as [21] and [25], and the more modern [15]. I recommend them to the interested reader. I hope that my current contribution might complement these and perhaps extend some of the ideas in them. The focus on E&M as a model theory for thinking about special relativity (relativistic covariance) and field theoretic manifestations of it is common at the graduate level, in books such as [26] and [20]. What I have tried to do is split off that discussion from the rest of the E&M problem-solving found in those books and amplify the structural elements. This book could be used alongside [19] for a second, advanced semester of E&M or as a standalone text for a course on classical field theory, one that might precede a quantum field theory course and whose techniques could be used fairly quickly in the quantum setting (much of the field–field interaction that quantum field theory is built to handle has classical analogues that benefit from many of the same techniques, including both perturbative ones and ones having to do with finding Green's functions).

Acknowledgments

My own view of field theories in physics was developed under the mentorship of Stanley Deser, and much of the motivation for exploring more complicated field theories by comparison/contrast with E&M comes directly from his approach to physics. My students and colleagues at Reed have been wonderful sounding boards for this material. In particular, I'd like to thank Owen Gross and David Latimer for commentary on drafts of this book; they have helped improve the text immensely. Irena Swanson in the math department gave excellent technical suggestions for the bits of complex analysis needed in the first appendix, and I thank her for her notes. Finally, David Griffiths has given numerous suggestions, in his inimitable manner, since I started the project – my thanks to him, as always, for helping me clarify, expand, and contract the text.