1 Special Relativity

1.1 Geometry

Special relativity focuses our attention on the geometry of space-time, rather than the usual Euclidean geometry of space by itself. We'll start by reviewing the role of rotations as a transformation that preserves a specific definition of length, then introduce Lorentz boosts as the analogous transformation that preserves a new definition of length.

1.1.1 Rotations

In two dimensions, we know it makes little difference (physically) how we orient the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axes – the laws of physics do not depend on the axis orientation. The description of those laws changes a little (what is "straight down" for one set of axes might be "off to the side" for another, as in Figure 1.1), but their fundamental predictions are independent of the details of these basis vectors.

A point in one coordinate system can be described in terms of a rotated coordinate system. For a vector that points from the origin to the point labeled by x and y in the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ basis vectors: $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$, we want a description of the same point in the coordinate system with basis vectors $\hat{\bar{\mathbf{x}}}$ and $\hat{\bar{\mathbf{y}}}$ – i.e., what are $\bar{x}$ and $\bar{y}$ in $\bar{\mathbf{r}} = \bar{x}\,\hat{\bar{\mathbf{x}}} + \bar{y}\,\hat{\bar{\mathbf{y}}}$? Referring to Figure 1.2, we can use the fact that the length of the vector $\mathbf{r}$ is the same as the length of the vector $\bar{\mathbf{r}}$ – lengths are invariant under rotation. If we call the length r, then:

$$\begin{aligned}\bar{x} &= r\,\cos(\psi - \theta) = r\,\cos\psi\,\cos\theta + r\,\sin\psi\,\sin\theta = x\,\cos\theta + y\,\sin\theta\\ \bar{y} &= r\,\sin(\psi - \theta) = r\,\sin\psi\,\cos\theta - r\,\cos\psi\,\sin\theta = y\,\cos\theta - x\,\sin\theta.\end{aligned} \tag{1.1}$$

So we know how to go back and forth between the barred coordinates and the unbarred ones. The connection between the two is provided by the invariant[1] length of the vector. In fact, the very definition of a rotation "transformation" between two coordinate systems is tied to length invariance. We could start with the idea that two coordinate systems agree on the length of a vector and use that to generate the transformation (1.1). Let's see how that goes: the most general linear transformation connecting $\bar{x}$ and $\bar{y}$ to x and y is

$$\begin{aligned}\bar{x} &= A\,x + B\,y\\ \bar{y} &= C\,x + D\,y\end{aligned} \tag{1.2}$$

[1] Invariant here means "the same in all coordinate systems related by some transformation."

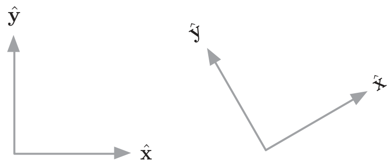

Fig. 1.1 Two different orientations for the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ basis vectors. In either case, the laws of physics are the same (e.g., $\mathbf{F} = m\,\mathbf{a}$ holds in both).

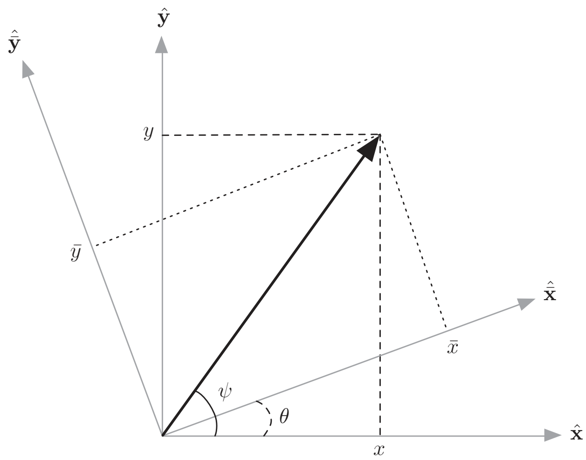

Fig. 1.2 The same point in two different bases (rotated by an angle θ with respect to one another). The angle ψ is the angle made by the vector with the $\hat{\mathbf{x}}$-axis.

for constants A, B, C, and D. Now, if we demand that $\bar{x}^2 + \bar{y}^2 = x^2 + y^2$, we have:

$$\left(A^2 + C^2\right)\, x^2 + \left(B^2 + D^2\right)\, y^2 + 2\,(A\,B + C\,D)\; x\,y = x^2 + y^2, \tag{1.3}$$

and then the requirement is: $A^2 + C^2 = 1$, $B^2 + D^2 = 1$, and $A\,B + C\,D = 0$. We can satisfy these with a one-parameter family of solutions by letting $A = \cos\theta$, $B = \sin\theta$, $C = -\sin\theta$, and $D = \cos\theta$ for arbitrary parameter θ. These choices reproduce (1.1) (other choices give back clockwise rotation).

1.1.2 Boosts

There is a fundamental invariant in special relativity, a quantity like length for rotations, that serves to define the transformation of interest. We'll start with the invariant and then build the transformation that preserves it. The length invariant used above for rotations comes from the Pythagorean notion of length. The invariant "length" in special relativity comes from the first postulate of the theory:

The speed of light is the same in all frames of reference traveling at constant velocity with respect to one another.

The idea, experimentally verified, is that no matter what you are doing, light has speed c. If you are at rest in a laboratory, and you measure the speed of light, you will get c. If you are running alongside a light beam in your laboratory, you will measure its speed to be c. If you are running toward a light beam in your laboratory, you will measure speed c. If the light is not traveling in some medium (like water), its speed will be c. This observation is very different from our everyday experience measuring relative speeds. If you are traveling at 25 mph to the right, and I am traveling next to you at 25 mph, then our relative speed is 0 mph. Not so with light.

What does the constancy of the speed of light tell us about lengths? Well, suppose I measure the position of light in my (one-dimensional) laboratory as a function of time: I flash a light on and off at time $t = 0$ at the origin of my coordinate system. Then the position of the light flash at time t is given by $x = c\,t$. Your laboratory, moving at constant speed with respect to mine, would call the position of the flash: $\bar{x} = c\,\bar{t}$ (assuming your lab lined up with mine at $t = \bar{t} = 0$ and $x = \bar{x} = 0$) at your time $\bar{t}$. We can make this look a lot like the Pythagorean length by squaring – our two coordinate systems must agree that:

$$-c^2\,\bar{t}^2 + \bar{x}^2 = -c^2\,t^2 + x^2 = 0. \tag{1.4}$$

Instead of $x^2 + y^2$, the new invariant is $-c^2\,t^2 + x^2$. There are two new elements here: (1) $c\,t$ is playing the role of a coordinate like x (the fact that c is the same for everyone makes it a safe quantity for setting units), and (2) the sign of the temporal portion is negative. Those aside, we have our invariant. While it was motivated by the constancy of the speed of light and applied in that setting, we'll now promote the invariant to a general rule (see Section 1.2.3 if this bothers you), and find the transformation that leaves the value $-c^2\,t^2 + x^2$ unchanged. Working from the most general linear transformation relating $c\,\bar{t}$ and $\bar{x}$ to $c\,t$ and x:

$$\begin{aligned} c\,\bar{t} &= A\,(c\,t) + B\,x \\ \bar{x} &= C\,(c\,t) + D\,x, \end{aligned} \tag{1.5}$$

and evaluating $-c^2\,\bar{t}^2 + \bar{x}^2$,

$$-(c\,\bar{t})^2 + \bar{x}^2 = -\left(A^2 - C^2\right)(c\,t)^2 + 2\,x\,(c\,t)\,(C\,D - A\,B) + \left(D^2 - B^2\right)x^2, \tag{1.6}$$

we can see that to make this equal to $-c^2\,t^2 + x^2$, we must have $A^2 - C^2 = 1$, $D^2 - B^2 = 1$, and $C\,D - A\,B = 0$. This is a lot like the requirements for length invariance above, but with funny signs. Noting that the hyperbolic version of $\cos^2\theta + \sin^2\theta = 1$ is

$$\cosh^2\eta - \sinh^2\eta = 1, \tag{1.7}$$

we can write $A = \cosh\eta$, $C = \sinh\eta$, $D = \cosh\eta$, and $B = \sinh\eta$. The transformation analogous to (1.1) is then:

$$\begin{aligned} c\,\bar{t} &= (c\,t)\,\cosh\eta + x\,\sinh\eta \\ \bar{x} &= (c\,t)\,\sinh\eta + x\,\cosh\eta. \end{aligned} \tag{1.8}$$

That's the form of the so-called Lorentz transformation. The parameter η, playing the role of θ for rotations, is called the "rapidity" of the transformation.

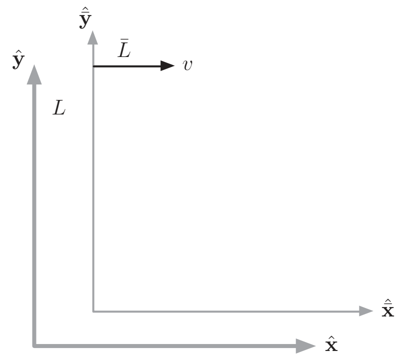

Fig. 1.3 A frame L and another frame, $\bar{L}$, moving to the right with speed v (in L). At time $t = 0$ in L, we have $\bar{t} = 0$ and $x = \bar{x} = 0$. Then at time t, the origin of $\bar{L}$ is at horizontal location $x = v\,t$ in L.

We can attach a physical picture to the transformation. Let the barred axes move with speed v to the right in the unbarred axes (the "lab frame," L). At time $t = 0$, we agree that $\bar{t} = 0$ and the spatial origins coincide ($x = \bar{x} = 0$, similar to keeping the origin fixed in rotations). Then the barred axes (the "moving frame," $\bar{L}$) have points $(c\,\bar{t}, \bar{x})$ related to the points in L by (1.8), and we need to find the relation between η and v. Referring to Figure 1.3, the origin of $\bar{L}$ is moving according to $x = v\,t$ in L. So we know to associate $\bar{x} = 0$ with $x = v\,t$, and using the second equation in (1.8) with this information gives:

$$0 = c\,t\,\sinh\eta + v\,t\,\cosh\eta \longrightarrow \tanh\eta = -\frac{v}{c}. \tag{1.9}$$

We can isolate both $\sinh\eta$ and $\cosh\eta$ given $\tanh\eta$ and the identity $\cosh^2\eta - \sinh^2\eta = 1$:

$$\sinh\eta = -\frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \cosh\eta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{1.10}$$

From these it is convenient to define the dimensionless quantities:

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \beta \equiv \frac{v}{c}, \tag{1.11}$$

and then (1.8) can be written:

$$\begin{aligned} c\,\bar{t} &= \gamma\left((c\,t) - x\,\beta\right) \\ \bar{x} &= \gamma\left(-(c\,t)\,\beta + x\right). \end{aligned} \tag{1.12}$$

This form of the "Lorentz boost" is equivalent to the hyperbolic cosine and sine transformation in (1.8) (which connect nicely with rotations), but written with the physical configuration (one lab traveling at constant speed with respect to another) in mind.

Problem 1.1 The definition of hyperbolic cosine is $\cosh\eta = \frac{1}{2}\left(e^{\eta} + e^{-\eta}\right)$, and hyperbolic sine is $\sinh\eta = \frac{1}{2}\left(e^{\eta} - e^{-\eta}\right)$. Using these definitions, show that $\cosh^2\eta - \sinh^2\eta = 1$.

Problem 1.2 Using the configuration shown in Figure 1.3, develop the "inverse Lorentz transformation" relating $c\,t$ and x to $c\,\bar{t}$ and $\bar{x}$ (inverting (1.12)). Hint: don't do any algebra.

Problem 1.3 Starting from $\tanh\eta = -\frac{v}{c}$, use the properties of hyperbolic sine and cosine to establish (1.10).

Problem 1.4 Show that $-c^2\,\bar{t}^2 + \bar{x}^2 = -c^2\,t^2 + x^2$ using the Lorentz transformations in "boost" form (1.12). If we take x–y coordinates (in addition to time), we would write the invariant as $s^2 \equiv -c^2\,t^2 + x^2 + y^2$. Show that rotations (1.1) (with $\bar{t} = t$) also preserve this "length"; i.e., show that $-c^2\,\bar{t}^2 + \bar{x}^2 + \bar{y}^2 = -c^2\,t^2 + x^2 + y^2$ where the spatial coordinates x and y are related by a rotation.

1.2 Examples and Implications

The Lorentz transformation is easy to describe and understand in terms of redefined "length," but its geometric implications for space and time are less familiar than their rotational analogues. For rotations, statements like "one lab's north is another's northwest" are perfectly reasonable, with no problem of interpretation. The inclusion of time in the transformation, allowing for relations like "one lab's time is another's space-and-time" sound more interesting but are really of the same basic type.

1.2.1 Length Contraction and Time Dilation

We can think of two "labs," L and $\bar{L}$, and compare observations made in each. We'll refer to the setup in Figure 1.3, so that the lab $\bar{L}$ moves at speed v to the right within L (and $x = \bar{x} = 0$ at $t = \bar{t} = 0$). As an example of moving measurements from one lab to the other: suppose we have a rod at rest in $\bar{L}$. Its length is $\Delta\bar{x}$ as measured in $\bar{L}$. What is the length of the rod in L? In order to make a pure length measurement, we will find Δx with $\Delta t = 0$, i.e., an *instantaneous*[2] length measurement in L. From the Lorentz transformation (1.12), we have:

$$\Delta\bar{x} = \gamma\,(-c\,\Delta t\,\beta + \Delta x) = \gamma\,\Delta x \longrightarrow \Delta x = \frac{1}{\gamma}\,\Delta\bar{x} = \sqrt{1 - \frac{v^2}{c^2}}\,\Delta\bar{x}. \tag{1.13}$$

For[3] $v < c$, we have $\Delta x < \Delta\bar{x}$, and the moving rod has a length in L that is *less than* its length in $\bar{L}$. We cryptically refer to this phenomenon as length contraction and say that "moving objects are shortened." To be concrete, suppose $v = \frac{3}{5}\,c$, then a meter stick in $\bar{L}$ ($\Delta\bar{x} = 1$ m) has length $\Delta x = \frac{4}{5}$ m in L.

What happens to a pure temporal measurement? Take a clock in $\bar{L}$ that is at the origin $\bar{x} = 0$ (and remains there). Suppose we measure an elapsed time in $\bar{L}$ that is $\Delta\bar{t}$, how much

[2] If you measure the ends of a moving rod at two different times, you will, of course, get an artificially inflated or deflated length.
[3] If $v > c$, $\Delta\bar{x}$ is imaginary, a reminder that superluminal motion is no longer sensible.

time elapsed in L? This time, we'll use the inverse Lorentz transformation (since we know $\Delta\bar{x} = 0$, the clock remained at the origin) from the solution to Problem 1.2:

$$c\,\Delta t = \gamma\left(c\,\Delta\bar{t} + \beta\,\Delta\bar{x}\right) = \gamma\,c\,\Delta\bar{t} \longrightarrow \Delta t = \gamma\,\Delta\bar{t}. \tag{1.14}$$

Remember that $\gamma > 1$ for $v < c$, so this time we have $\Delta t > \Delta\bar{t}$: if 1 s passes in $\bar{L}$, traveling at $v = \frac{3}{5}\,c$, then $\Delta t = \frac{5}{4}$ s passes in L. The phrase we use to describe this "time dilation" is "moving clocks run slow."

Time dilation and length contraction are very real physical effects. The muon, an elementary particle, has a lifetime of 2×10^{-6} s when at rest. Muons produced in the upper atmosphere travel very fast. For $v = .999\,c$, we would expect a muon to decay in about 600 m, yet they make it down to the surface of the earth for detection. The reason is time dilation. From the earth's point of view, the muon's lifetime is $T = 1/\sqrt{1 - 0.999^2}\left(2 \times 10^{-6}\text{ s}\right) \approx 4 \times 10^{-5}$ s. In this amount of time, they can go a distance $\approx 0.999\,c\,T \approx 12$ km, far enough to reach the earth's surface.

1.2.2 Directions Perpendicular to Travel

The "first" postulate of special relativity is the following.

The laws of physics are the same in all inertial frames of reference

or, sloppily, in any frames moving at constant velocity with respect to each other. That doesn't have a lot of quantitative meaning just yet, but it can be used to establish that, for example, length contraction occurs only in the direction parallel to the motion. When we define the Lorentz boost in three dimensions for a lab $\bar{L}$ moving along the $\hat{\mathbf{x}}$-axis in L, we say that $(c\,t)$, x, y, and z become, in the "barred" coordinates:

$$\begin{aligned} c\,\bar{t} &= \gamma\,((c\,t) - x\,\beta) \\ \bar{x} &= \gamma\,(-(c\,t)\,\beta + x) \\ \bar{y} &= y \\ \bar{z} &= z. \end{aligned} \tag{1.15}$$

But how do we *know* that nothing interesting happens in the y- and z-directions (perpendicular to the relative motion)? We can use the first postulate – suppose I have a door that is exactly my height, so that I just barely fit under it at rest as in the top panel of Figure 1.4. Now assume that length contraction holds in the $\hat{\mathbf{y}}$-direction even for relative motion in the $\hat{\mathbf{x}}$-direction. I move toward the door at speed v. From the door's point of view, I am moving and by our (false) assumption, I appear shorter than normal (to the door) and easily make it through the door. From my point of view, the door is moving toward me; it appears shorter than when it is at rest, and I do not make it through. This conflict violates the notion that the door and I had better agree on whether I can get through. The easy fix: there is no length contraction perpendicular to the direction of travel.

Problem 1.5 A meter stick sits at rest in $\bar{L}$. The $\bar{L}$ frame is moving to the right at speed v in the lab (the same setup as shown in Figure 1.3). In L, the stick is measured to have length $12/13$ m. What is v?

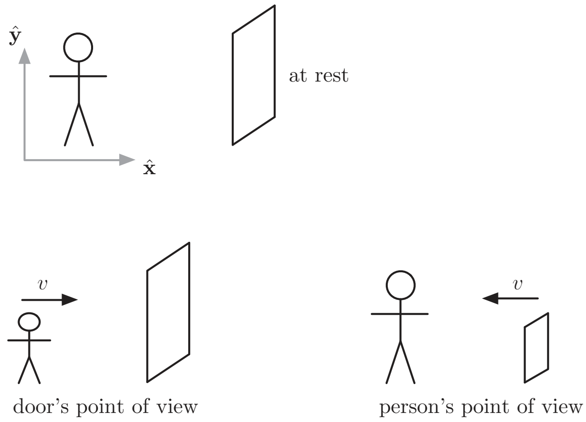

Fig. 1.4 A person and a door are the same height when at rest (top). If length contraction happened in directions perpendicular to travel, we'd have a shorter person from the door's point of view (bottom left) or a shorter door from the person's point of view (bottom right). These two points of view lead to different physical predictions: the person either makes it through the door or does not, and we can't have that. There must therefore be no length contraction in the direction perpendicular to the relative motion.

Problem 1.6 A clock moves through a lab with speed $v = (12/13)\,c$. It goes from $x = 0$ to $x = 5$ m. How long did its trip take in the lab? How long did its trip take in the rest frame of the clock (i.e., in $\bar{L}$ moving at speed v, where the clock remains at $\bar{x} = 0$)?

Problem 1.7 You take off in a rocket ship, headed for the stars, at a speed of $v = 3/5\,c$. After a year of traveling, you turn around and head back to earth at the same speed. How much time has passed on earth? (Make $\bar{L}$ the ship's frame.)

1.2.3 Alternative

We generated the Lorentz transformations by defining the quantity:

$$s^2 \equiv -c^2\,t^2 + x^2 + y^2 + z^2 \tag{1.16}$$

and demanding that it be unchanged by the transformation. But our only example, light traveling at c, has $s^2 = 0$, so it is still possible that we should allow, for $s^2 \neq 0$, a scaling. Maybe we require only that:

$$\bar{s}^2 = \alpha^2\,s^2, \tag{1.17}$$

where $\bar{s}^2$ is in the transformed frame and α is a scaling factor. If $\bar{s}^2 = s^2 = 0$, we wouldn't see the α at all. We will now show that the constancy of the speed of light requires $\alpha = \pm 1$, and we'll take the positive root to avoid switching the signs of intervals. First note that α cannot be a constant function of the speed v (other than 1) in the transformation. If it were, we would get $\bar{s}^2 \neq s^2$ in the $v = 0$ limit. So $\alpha(v)$ has the constraint that $\alpha(0) = 1$ in order to recover the same s^2 when the boost speed is zero.

Now the Lorentz transformation that puts an α^2 in front of the length s^2 upon transformation, without violating any of our other rules, is:

$$\begin{aligned} c\,\bar{t} &= \alpha\,\gamma\,((c\,t) - x\,\beta) \\ \bar{x} &= \alpha\,\gamma\,(-(c\,t)\,\beta + x) \\ \bar{y} &= y \\ \bar{z} &= z, \end{aligned} \tag{1.18}$$

for a boost in the x-direction. The inverse transformation can be obtained by taking $\beta \rightarrow -\beta$ (as usual) and $\alpha \rightarrow 1/\alpha$.

In $\bar{L}$, we shoot a laser along the $\hat{\bar{\mathbf{y}}}$-axis. In a time $\Delta\bar{t}$, the light has traveled $\Delta\bar{x} = \Delta\bar{z} = 0$, $\Delta\bar{y} = c\,\Delta\bar{t}$. Using these in the inverse transformation associated with (1.18), we have

$$\Delta t = \frac{\gamma}{\alpha}\,\Delta\bar{t}, \qquad \Delta x = \frac{\gamma}{\alpha}\,v\,\Delta\bar{t}, \qquad \Delta y = c\,\Delta\bar{t}, \tag{1.19}$$

with $\Delta z = 0$. The distance traveled by the light in L is:

$$\Delta \equiv \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\frac{\gamma^2}{\alpha^2}\,v^2 + c^2}\,\Delta\bar{t} \tag{1.20}$$

so that the speed of the light, as measured in L, is

$$c_L = \frac{\Delta}{\Delta t} = \sqrt{v^2 + \alpha^2\,c^2\left(1 - \frac{v^2}{c^2}\right)} = \sqrt{v^2\left(1 - \alpha^2\right) + c^2}, \tag{1.21}$$

and the only way for the speed of light to be c in L, $c_L = c$, is if $\alpha = \pm 1$, and we'll take the positive root.

1.2.4 Simultaneity

Suppose we have an "event" that occurs in the lab at position x_1, time t_1 (e.g., a firecracker goes off) and another event that occurs at x_2, time t_1. The two events are separated spatially but happen at the same time. For our moving $\bar{L}$ coordinate system, these points become:

$$\begin{aligned} c\,\bar{t}_1 &= \gamma\,((c\,t_1) - x_1\,\beta)\,, \qquad \bar{x}_1 = \gamma\,(x_1 - (c\,t_1)\,\beta) \\ c\,\bar{t}_2 &= \gamma\,((c\,t_1) - x_2\,\beta)\,, \qquad \bar{x}_2 = \gamma\,(x_2 - (c\,t_1)\,\beta)\,, \end{aligned} \tag{1.22}$$

so that $\bar{t}_2 - \bar{t}_1 = -\frac{\gamma\,\beta}{c}\,(x_2 - x_1) \neq 0$. The two events did not occur at the same time in $\bar{L}$. Events simultaneous in one coordinate system are not simultaneous in another.

Related to simultaneity is the idea of "causality" – one event (at t_1, x_1, say) can cause another (at t_2, x_2) only if there is a way for a signal traveling at the speed of light (or less) to be sent between the two in a time less than (or equal to) $t_2 - t_1$. Think of E&M, where electric and magnetic fields propagate (in vacuum) at speed c. The only way for one charge to act on another electromagnetically is for the fields to travel from one location to the other at speed c. The requirement that two space-time points be "causally connected"[4] amounts to: $c^2\,(t_2 - t_1)^2 \geq (x_2 - x_1)^2$ (where we squared both sides to avoid absolute values), or

$$-c^2\,(t_2 - t_1)^2 + (x_2 - x_1)^2 \leq 0. \tag{1.23}$$

Problem 1.8 Two events (t_1 at x_1 and t_2 at x_2) are causally related if $-c^2\,(t_2 - t_1)^2 + (x_2 - x_1)^2 \leq 0$. Show that if event one causes event two in L (i.e., the two events are causally related with $t_1 < t_2$), then event one causes event two in any $\bar{L}$ traveling at speed v in L (where L and $\bar{L}$ are related as usual by Figure 1.3).

1.2.5 Minkowksi Diagrams

Just as we can draw motion in the "usual" two-dimensional space spanned by $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, and relate that motion to rotated (or otherwise transformed) axes, we can think of motion occurring in the two-dimensional space spanned by time $c\,\hat{\mathbf{t}}$ and $\hat{\mathbf{x}}$.[5] Referring to Figure 1.5, we can display motion that takes the form $x(t)$ by inverting to get $t(x)$ and then plotting $c\,t(x)$ versus x. Light travels with $x = c\,t$, and the dashed line in Figure 1.5 shows light emitted from the origin. For a particle traveling with constant speed v, we have $x = v\,t$ and can invert to write

$$c\,t = \frac{c}{v}\,x, \tag{1.24}$$

describing a line with slope c/v. For light, the slope is 1, and for particle motion with $v < c$, the slope of the line is > 1. Superluminal motion occurs with lines of slope < 1. For the "fun" trajectory shown in Figure 1.5, if we look at the tangent to a point on the curve, we can estimate the speed of the particle at a given time – portions of that trajectory do have tangent lines with slope < 1, indicating that the particle is traveling faster than the speed of light.

For a particle traveling with constant speed through "the lab" L, we know that the particle is at rest in its own $\bar{L}$ (moving with constant speed through L). That means that the line of the particle's motion in L represents the particle's $c\,\hat{\bar{\mathbf{t}}}$-axis (purely temporal motion). What does the $\hat{\bar{\mathbf{x}}}$-axis look like? Well, those are points for which $\bar{t} = 0$. From the inverse of the transformation (1.12), we have

$$\begin{aligned} c\,t &= \gamma\,\beta\,\bar{x} \\ x &= \gamma\,\bar{x}, \end{aligned} \tag{1.25}$$

defining the line: $c\,t = \frac{v}{c}\,x$ in L. That's just the reflection of the line $c\,t = \frac{c}{v}\,x$ about the "light" line. In Figure 1.6, we have the picture relating the axes for boosts analogous to Figure 1.1 for rotations. By looking at the two coordinate systems, it should be clear that a pure temporal interval in the barred coordinates, $\Delta\bar{t}$, corresponds to a combination of Δt and Δx in the unbarred coordinate system (and vice versa).

It is interesting to note that our new "Minkowski length" definition: $s^2 = -c^2\,t^2 + x^2$ allows for three distinct options: s^2 can be greater than, less than, or equal to zero. We

[4] Meaning that light can travel between the two points in a time that is less than their temporal separation.
[5] The basis vector for the temporal direction is denoted $c\,\hat{\mathbf{t}}$ because writing $\hat{ct}$ looks awkward.

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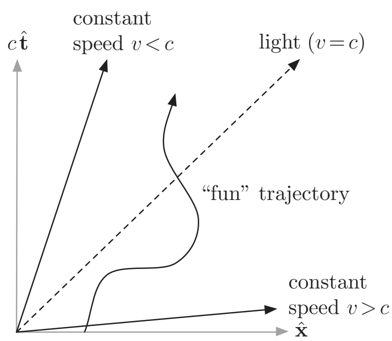

Fig. 1.5 A Minkowski diagram: we plot the motion of a particle or light with $c\,t$ as the vertical axis and x as the horizontal one.

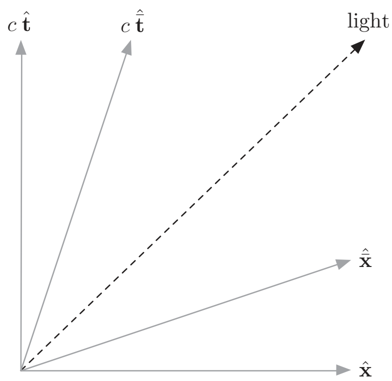

Fig. 1.6 Two coordinate systems related by a Lorentz boost (1.12).

generated interest in this particular combination of $c\,t$ and x by thinking about light, for which $-c^2\,t^2 + x^2 = 0$ (since light travels along the line $x = c\,t$). We call points separated by "zero" light-like since a light-like trajectory connects these points. But we also have motion with $v < c$: $x = v\,t$, and this gives a separation $s^2 = -c^2\,t^2 + v^2\,t^2 < 0$. Such intervals are called "time-like" because there is a rest frame (in which only temporal motion occurs) connecting the two points. Finally, for $v > c$, $-c^2\,t^2 + x^2 = -c^2\,t^2 + v^2\,t^2 > 0$ and these separations are "space-like" (like motion along the spatial $\hat{\mathbf{x}}$ and $\hat{\bar{\mathbf{x}}}$ axes). For material particles, only time-like separations are possible. It is clear that speeds greater than c cannot be achieved: the boost factor γ becomes imaginary. That's a mathematical warning sign, the physical barrier is the amount of energy that a massive particle moving at c would have (infinite, as we shall see).

Problem 1.9 We have $\bar{L}$ moving at constant speed v in L as in Figure 1.3 (with origins $x = \bar{x} = 0$ that overlap at $t = \bar{t} = 0$). In $\bar{L}$, we observe the following two events: $\bar{t}_1 = 0$, $\bar{x}_1 = 0$, and $\bar{t}_2 = 0$, $\bar{x}_2 = d$. The $c\,\hat{\bar{\mathbf{t}}}$–$\hat{\bar{\mathbf{x}}}$ picture is shown in Figure 1.7. Find the location and time of these two events in L and draw the $c\,\hat{\mathbf{t}} - \hat{\mathbf{x}}$ version of the picture using $d = 1$ m and $v = 4/5$ c.