# 1 Introduction

Dynamics is traditionally defined as the classical study of motion with respect to the physical causes of motion, that is, forces and moments. Kinematics, on the other hand, is concerned with the study of motion *without* respect to the underlying physical causes. In this sense, kinematics is really a fundamental prerequisite upon which dynamics is constructed.

For the purposes of this text, the terms *dynamics* and *mechanics* are taken to be synonymous. The choice of which term is used is based more on the academic community than on a strict technical distinction. The engineering community typically adopts the term *dynamics* and the physics and applied mathematics communities typically adopt the term *mechanics*. The term *dynamics* is predominantly used in this text.

## 1.1 Historical Background

Interest in the dynamics of linked multibody systems has existed throughout much of recorded human history. As an example, representations of the human form in art have included anthropomorphic constructions made up of mechanical elements like those depicted in Giovanni Braccelli's *Bizzarie di Varie Figure* published in 1624 (see Figure 1.1). Braccelli's art coincided with the birth of the mechanical philosophy of René Descartes, Pierre Gassendi, and others. The mechanical philosophy sought to describe physical phenomena in terms of intricate mechanisms. Decades after the birth of the mechanical philosophy, a systematic theory of mechanics began to flourish with Newtonian mechanics.

Analytical dynamics (historically referred to as *analytical mechanics*) is identified with a number of formulations of classical mechanics that arose after Isaac Newton published his *Philosophiae Naturalis Principia Mathematica* in 1687. The cornerstones of Newtonian mechanics are his laws of motion, which were applied to point masses. From a modern perspective, Newton's second law can be seen more generally as a conservation law, specifically, as a law of conservation of momentum. As such, Newtonian mechanics, and its extension to extended (rigid) bodies by Leonhard Euler, are based on two fundamental conservation principles: (1) the conservation of translational momentum and (2) the conservation of angular momentum.

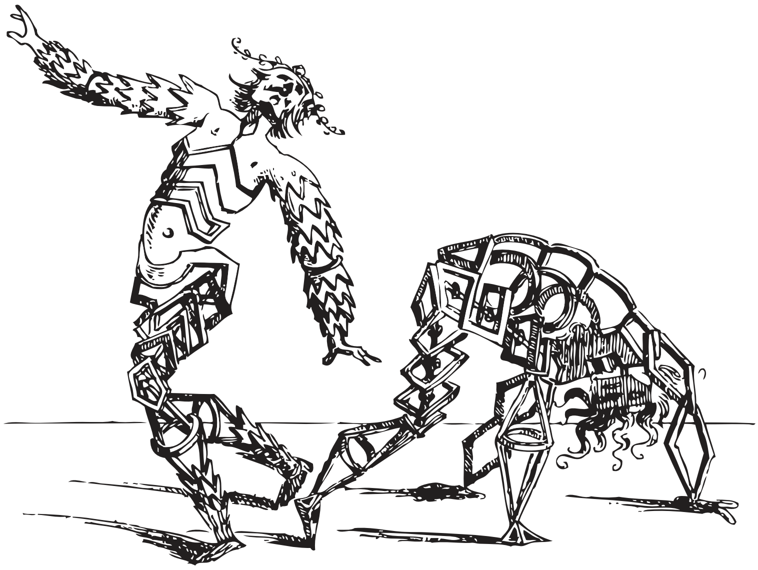

**Figure 1.1** Anthropomorphic forms adapted from plate no. 12 of *Bizzarie di Varie Figure*, 1624, by Giovanni Battista Braccelli, Livorno, Library of Congress Lessing J. Rosenwald Collection. Braccelli's Mannerist-styled work consists of a set of 50 etchings depicting anthropomorphic mechanical figures like this one. The human form has long been an inspiration in the study of mechanical systems.

In contrast to the conservation principles underlying the Newton-Euler mechanics, the analytical dynamics that followed were based on variational principles. The conservation of vector quantities like translational and angular momentum was replaced with principles rooted in the variation of scalar quantities like work and energy. An important precursor to the subsequent development of variational mechanics was the concept of the *vis viva* (living force) proposed by Gottfried Leibniz, a contemporary of Newton. The *vis viva* corresponds to our present notion of kinetic energy. In Leibniz's mechanics, momentum was replaced by kinetic energy and force was replaced by work of the force (Lanczos 1986).

Development of variational mechanics required a mathematical tool beyond the basic calculus of Newton and Leibniz. The calculus of variations, concerned with extremizing functionals (mappings of functions to scalar values), emerged as this tool. Johann Bernoulli was the first to exploit the calculus of variations in solving the brachistochrone curve problem. However, Euler is usually credited with the formal development of the calculus of variations in his 1744 *Methodus inveniendi*.

Around the time of Euler's *Methodus inveniendi*, in 1743, Jean Le Rond d'Alembert published his *Traité de dynamique*. This articulated d'Alembert's Principle of Virtual Work. Although Johann Bernoulli is credited with first proposing the Principle of Virtual Work for cases of static equilibrium, d'Alembert is credited with extending the principle to dynamic equilibrium by interpreting the acceleration terms in Newton's equation of motion as inertial forces. The principle is based on the notion of virtual displacement, defined as an infinitesimal change of the system's configuration coordinates while time is frozen. The displacement is virtual because no actual displacement occurs. Rather, a virtual displacement is used as a conceptual mechanism. D'Alembert's Principle can be viewed as a *zeroth*-order variational principle, as it is based on the *zeroth*-order derivative of displacement.

It should be noted that with the advent of analytical dynamics and variational principles like d'Alembert's, the concept of generalized coordinates became relevant. As the name implies, these coordinates are a generalization of the Cartesian coordinates used in Newton-Euler mechanics, whereby any consistent set of parameters that uniquely describe the configuration of the system can be chosen. The vector space defined by these generalized coordinates forms the configuration space of the system.

Following d'Alembert's Principle, the monumental work of Joseph-Louis Lagrange (born Giuseppe Lodovico Lagrangia) resulted in what is now known as Lagrangian mechanics. Together with contributions from Euler and William Rowan Hamilton, the Euler-Lagrange equations emerged as a logical consequence of Hamilton's Principle of Least Action. Disentangling the individual contributions of these three eminent mechanicians with respect to Lagrangian mechanics can be a bit tedious. Consequently, we will not proceed in chronological order when discussing this.

Hamilton's Principle of Least Action has been referred to as "the most direct and most natural transformation of d'Alembert's into a minimum principle" (Lanczos 1986, p. 111). The principle states that the path of a system in configuration space during a time interval is such that the action is stationary under all path variations. The action is defined as the integral of the Lagrangian over the time interval, where the Lagrangian is defined as the difference between the kinetic and potential energies of the system. The Euler-Lagrange equations of motion for the system emerge directly from this principle. Lagrange published his seminal work, *Mécanique analytique*, in 1788, formalizing these ideas into what is now known as Lagrangian mechanics.

Hamilton's reformulation of Lagrangian mechanics, published in 1833, constitutes what is now known as Hamiltonian mechanics. This reformulation involves a transformation of the Euler-Lagrange equations from a set of second-order differential equations in the generalized coordinates to a set of first-order differential equations in the generalized coordinates and generalized momenta. The Hamiltonian is defined as a new invariant corresponding to the total energy. Hamilton's equations written in terms of the Hamiltonian are known as Hamilton's canonical equations.

Up to this point, the historical flow of the variational principles of mechanics has followed the path from d'Alembert's Principle to Hamilton's Principle to the Euler-Lagrange equations and, finally, to Hamilton's canonical equations. It was mentioned that d'Alembert's Principle can be viewed as a *zeroth*-order variational principle. Holonomic constraints, which take the form of algebraic functions of the generalized coordinates and possibly time, can be inherently addressed by *zeroth*-order variational principles using the method of Lagrange multipliers.

Higher-order variational principles have also been proposed. One of the advantages of higher-order variational principles is the ability to address nonholonomic constraints, which take the form of algebraic functions of the higher-order derivatives of the

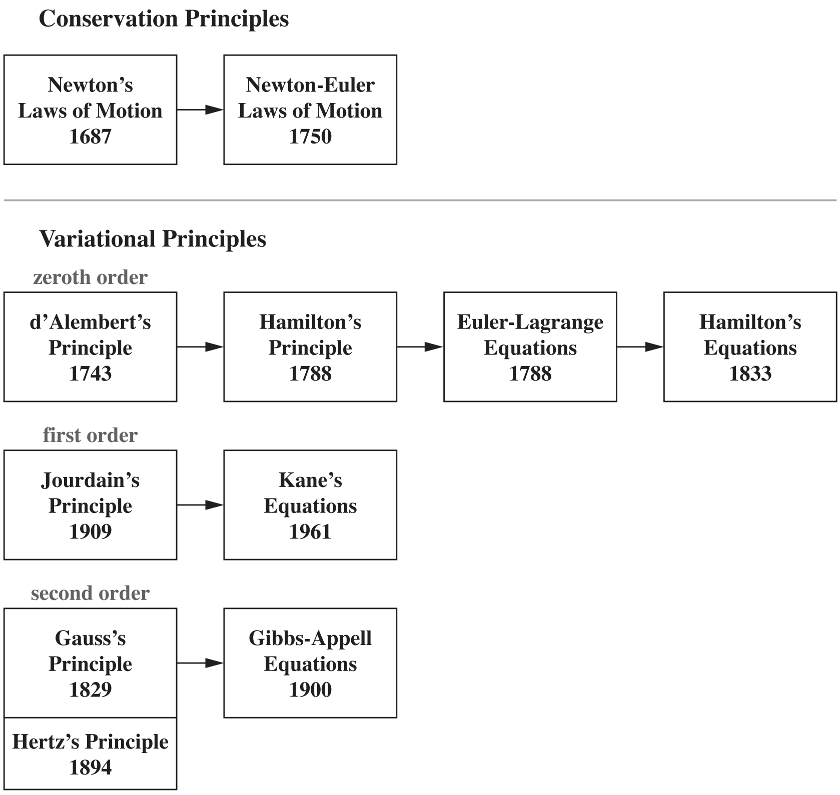


**Figure 1.2** Historical progression of key principles of analytical dynamics. The variational principles, which form the basis of analytical dynamics, are comprised of a number of formulations that arose after Newton's mechanics.

generalized coordinates (generalized velocities and accelerations) and possibly time. The first-order variational principle, published by Philip E. B. Jourdain in 1909, is based on the notion of virtual velocity. Virtual power assumes the role that virtual work assumes in d'Alembert's Principle. Subsequent developments by Thomas Kane in 1961 essentially rediscovered Jourdain's Principle. Kane's approach extended Jourdain's approach to rigid bodies and introduced quasi-velocities to implicitly handle nonholonomic constraints.

Jourdain's Principle was influenced by Carl Friedrich Gauss's second-order variational principle, based on the notion of virtual accelerations, published in 1829. Gauss used the notion of virtual acceleration to establish a true minimum principle known as Gauss's Principle of Least Constraint, which minimizes the quadratic form known as the Gauss function. Heinrich Rudolf Hertz reinterpreted a special case of Gauss's Principle as the Principle of Least Curvature in 1894. Subsequent developments by Josiah Willard Gibbs (1879), Paul Appell (1900), and others led to the Gibbs-Appell equations, which have their lineage in Gauss's Principle. The Gibbs-Appell equations, like Kane's equations, make use of quasi-variables – in this case quasi-accelerations – to implicitly handle nonholonomic constraints. Although they derive from variational principles of different order, the Gibbs-Appell equations and Kane's equations can be viewed as identical. However, because the Gibbs-Appell equations are derived from a second-order variational principle, they can be stated in a concise form as the gradient of a scalar function, known as the Gibbs function, with respect to the quasi-accelerations. The Gibbs function can be related to the Gauss function.

Figure 1.2 summarizes the historical progression of some of the key principles of analytical dynamics that are covered in this book. I have by no means presented an exhaustive history, and the interested reader is referred to Dugas's excellent history of mechanics (Dugas 1988), which addresses ancient through modern developments in mechanics.

## 1.2 Devices That Illustrate Principles of Analytical Dynamics

Before jumping into a formal exposition of analytical dynamics, we will look at some motivating examples. Toys and other objects of amusement tend to make the most compelling examples. The reader is encouraged to refer back to these when encountering similar detailed technical examples presented in the subsequent chapters.

Figure 1.3 displays some devices composed of branching kinematic chains. A modified double pendulum (top), the Swinging Sticks Kinetic Energy Sculpture by BTS Trading GmbH, exhibits chaotic motion characterized by a sensitive dependence on initial conditions. As with all real-world mechanical systems, it dissipates energy; however, in this example, the double pendulum gives the illusion of perpetual motion through the use of electromagnetic coils mounted in the base and permanent neodymium rare-earth magnets mounted in the arms. The electromagnets measure the speed of the rotating arms and impart additional kinetic energy into the system. The gimbaled Super Precision Gyroscope distributed by Gyroscope.com (bottom left) consists of a high-speed rotor mounted, in this case, on a two-axis gimbal. The rotor on a high-speed gyroscope is precisely balanced and mounted on low-friction bearings to demonstrate the conservation of angular momentum (see the example in Section 5.3.5). Similar to the Swinging Sticks Kinetic Energy Sculpture is the Chaos Machine by Fat Brain Toys, a reconfigurable tree-structured mechanism (bottom right). This simple device illustrates the complex motion characteristic of multilink kinematic chains.

Figure 1.4 displays some devices that operate under holonomic constraints. These constraints are discussed in detail in later chapters. The holonomically constrained devices shown here all involve loop closures that can be represented as algebraic conditions on the configuration coordinates. The Falcon (top), by Novint Technologies Inc., is a haptic (force feedback) game controller based on the kinematics of the Delta parallel robot (Clavel 1991). Three translational degrees of freedom are provided by the kinematic structure of the Falcon, which uses four-bar parallelogram linkages in the three arms to maintain the fixed orientation of the end effector. Three actuators mounted in

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**Figure 1.3** Some devices composed of branching kinematic chains. (Top) The Swinging Sticks Kinetic Energy Sculpture by BTS Trading GmbH. (Bottom Left) A gimbaled Super Precision Gyroscope distributed by Gyroscope.com. (Bottom Right) The Chaos Machine by Fat Brain Toys. All images © 2016 Vincent De Sapio.

the base allow the device to provide force feedback to the user. The Hoberman Sphere (bottom left) is a collapsing spherical structure made up of six rings, each of which comprises a series of connected four-bar parallelogram linkages that produce scissor-like motion. The overall structure has 1 degree of freedom and is able to radially expand and contract. A Stirling engine (bottom right), produced by Wiggers Stirling HeiBluft Modellbau, makes use of closed chain slider-crank and four-bar linkages to convert reciprocating piston motion into rotational motion of a flywheel.

Figure 1.5 displays some devices that operate under nonholonomic constraints. As with holonomic constraints, we discuss nonholonomic constraints in detail in later chapters. The nonholonomically constrained devices shown here all involve rolling/spinning constraints that can be represented as algebraic conditions on the configuration

**Figure 1.4** Some devices that operate under holonomic constraints. (Top) The Falcon haptic game controller by Novint Technologies Inc. (Bottom Left) The Hoberman Sphere © 1993 Charles Hoberman. (Bottom Right) A Stirling engine by Wiggers Stirling HeiBluft Modellbau. All images © 2016 Vincent De Sapio.

velocities. Specifically, the no-slip rolling/spinning condition requires *zero* velocity of the instantaneous material contact point of the device with the external surface. The Euler's Disk (top), distributed by Toysmith, illustrates the dynamics of a rolling/spinning disk on a flat surface. The complex motion of the disk produces intricate traces, which can be investigated in simulation (see an example in Section 6.2.5). Ollie (bottom left), by Sphero Inc., is a two-wheeled robot controlled by a smartphone app. The wheels are independently driven to allow steering and maneuvering. A gyroscope and accelerometer are incorporated into the control unit for inertial sensing. Sphero SPRK edition (bottom right), by Sphero Inc., is a spherical robot also controlled by a smartphone app. Internal drive wheels and a stabilizer provide forward propulsion and maneuvering. As with Ollie, a gyroscope and accelerometer are incorporated into the control unit for inertial sensing.

**Figure 1.5** Some devices that operate under nonholonomic constraints. (Top) Euler's Disk by Toysmith. (Bottom Left) Ollie by Sphero Inc. (Bottom Right) Sphero SPRK edition by Sphero Inc. All images © 2016 Vincent De Sapio.

## 1.3 Scope of This Book

This book is intended to cover the foundations of analytical dynamics of discrete systems. By discrete systems, we mean systems made up of a discrete set of point masses or rigid bodies, as opposed to continuous systems. Continuous systems are the subject of continuum mechanics, which addresses infinite-dimensional deformable bodies, that is, bodies with infinite degrees of freedom. The study of flexible multibody systems, which include both rigid and deformable bodies, is an active area of research. Although such systems are outside the scope of the main body of this book, the appendix addresses some of the basics of the application of continuum mechanics to flexible multibody systems.

The material covered in this book is intended for an an intermediate to advanced graduate-level audience. As such, it is not intended as an introductory book on dynamics or classical mechanics. Some of the topics covered, particularly higher-order variational principles, are not commonly covered in engineering dynamics texts. Preceding the material on dynamics is a chapter providing a brief mathematical background in linear algebra, vectors and tensors, differential geometry, and optimization. A chapter on the kinematics of discrete systems directly precedes the chapters devoted to dynamics. Conservation principles are then addressed, followed by variational principles. Separate chapters cover zeroth-, first-, and second-order principles.

The variational principles are based on a configuration space description comprising generalized coordinates. After addressing these configuration space formulations, we present a chapter based on an alternate, task space formulation of dynamics using task coordinates. The following chapter presents applications to biomechanical systems. Such systems are an active area of study, and this chapter is intended to provide a brief introduction to system-level modeling of musculoskeletal and neuromuscular dynamics. The final chapter provides a short survey of some analytical dynamics software. This includes examples of general purpose mathematical software as well as dedicated multibody dynamics software.

The subsequent chapters provide example problems that are worked out in detail. The intention is to give the reader exposure to systematic approaches to applying the concepts presented to practical examples, thereby reinforcing abstract concepts with concrete applications.