

Quantum Information Theory

Developing many of the major, exciting pre- and post-millennium developments from the ground up, this book is an ideal entry point for graduate students into quantum information theory. Significant attention is given to quantum mechanics for quantum information theory, and careful studies of the important protocols of teleportation, super-dense coding, and entanglement distribution are presented.

In this new edition, readers can expect to find over 100 pages of new material, including detailed discussions of Bell's theorem, the CHSH game, Tsirelson's theorem, the axiomatic approach to quantum channels, the definition of the diamond norm and its interpretation, and a proof of the Choi–Kraus theorem. Discussion of the importance of the quantum dynamic capacity formula has been completely revised, and many new exercises and references have been added. This new edition will be warmly welcomed by the upcoming generation of quantum information theorists and the already-established community of classical information theorists.

MARK M. WILDE is an Assistant Professor in the Department of Physics and Astronomy, and in the Center for Computation and Technology, at Louisiana State University in Baton Rouge, Louisiana. He is the recipient of a National Science Foundation Career Development Award and the APS-IUSSTF Professorship Award in Physics. He is also a Senior Member of the IEEE, and is currently serving as Associate Editor for Quantum Information Theory for the leading journal, *IEEE Transactions on Information Theory*. His current research interests are in quantum Shannon theory, quantum optical communication, quantum computational complexity theory, and quantum error correction.

Cambridge University Press

978-1-107-17616-4 – Quantum Information Theory

Mark M. Wilde

Frontmatter

[More Information](#)

Quantum Information Theory

Second Edition

MARK M. WILDE
Louisiana State University



CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India
79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107176164

© Mark M. Wilde 2013, 2017

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2013

Second edition 2017

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

ISBN 978-1-107-17616-4 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party Internet Web sites referred to in this publication and does not guarantee that any content on such Web sites is, or will remain, accurate or appropriate.

Contents

	<i>Preface to the Second Edition</i>	<i>page</i> xi
	<i>Preface to the First Edition</i>	xiii
	<i>How To Use This Book</i>	xv
Part I	Introduction	1
1	Concepts in Quantum Shannon Theory	3
	1.1 Overview of the Quantum Theory	7
	1.2 The Emergence of Quantum Shannon Theory	11
2	Classical Shannon Theory	26
	2.1 Data Compression	26
	2.2 Channel Capacity	35
	2.3 Summary	49
Part II	The Quantum Theory	51
3	The Noiseless Quantum Theory	53
	3.1 Overview	54
	3.2 Quantum Bits	54
	3.3 Reversible Evolution	60
	3.4 Measurement	67
	3.5 Composite Quantum Systems	73
	3.6 Entanglement	81
	3.7 Summary and Extensions to Qudit States	93
	3.8 Schmidt Decomposition	98
	3.9 History and Further Reading	100
4	The Noisy Quantum Theory	101
	4.1 Noisy Quantum States	102
	4.2 Measurement in the Noisy Quantum Theory	114
	4.3 Composite Noisy Quantum Systems	117
	4.4 Quantum Evolutions	129
	4.5 Interpretations of Quantum Channels	141
	4.6 Quantum Channels are All-Encompassing	143
	4.7 Examples of Quantum Channels	154

vi	Contents		
	4.8	Summary	162
	4.9	History and Further Reading	163
5	The Purified Quantum Theory		164
	5.1	Purification	165
	5.2	Isometric Evolution	168
	5.3	Coherent Quantum Instrument	179
	5.4	Coherent Measurement	180
	5.5	History and Further Reading	181
	Part III	Unit Quantum Protocols	183
6	Three Unit Quantum Protocols		185
	6.1	Non-Local Unit Resources	186
	6.2	Protocols	188
	6.3	Optimality of the Three Unit Protocols	197
	6.4	Extensions for Quantum Shannon Theory	199
	6.5	Three Unit Qudit Protocols	200
	6.6	History and Further Reading	205
7	Coherent Protocols		207
	7.1	Definition of Coherent Communication	208
	7.2	Implementations of a Coherent Bit Channel	210
	7.3	Coherent Dense Coding	211
	7.4	Coherent Teleportation	213
	7.5	Coherent Communication Identity	215
	7.6	History and Further Reading	216
8	Unit Resource Capacity Region		217
	8.1	The Unit Resource Achievable Region	217
	8.2	The Direct Coding Theorem	221
	8.3	The Converse Theorem	222
	8.4	History and Further Reading	226
	Part IV	Tools of Quantum Shannon Theory	227
9	Distance Measures		229
	9.1	Trace Distance	230
	9.2	Fidelity	243
	9.3	Relations between Trace Distance and Fidelity	253
	9.4	Gentle Measurement	257
	9.5	Fidelity of a Quantum Channel	260
	9.6	The Hilbert–Schmidt Distance Measure	265
	9.7	History and Further Reading	266

10	Classical Information and Entropy	267
10.1	Entropy of a Random Variable	268
10.2	Conditional Entropy	273
10.3	Joint Entropy	274
10.4	Mutual Information	275
10.5	Relative Entropy	276
10.6	Conditional Mutual Information	278
10.7	Entropy Inequalities	279
10.8	Near Saturation of Entropy Inequalities	291
10.9	Classical Information from Quantum Systems	297
10.10	History and Further Reading	299
11	Quantum Information and Entropy	300
11.1	Quantum Entropy	301
11.2	Joint Quantum Entropy	306
11.3	Potential yet Unsatisfactory Definitions of Conditional Quantum Entropy	309
11.4	Conditional Quantum Entropy	311
11.5	Coherent Information	313
11.6	Quantum Mutual Information	315
11.7	Conditional Quantum Mutual Information	318
11.8	Quantum Relative Entropy	321
11.9	Quantum Entropy Inequalities	327
11.10	Continuity of Quantum Entropy	339
11.11	History and Further Reading	345
12	Quantum Entropy Inequalities and Recoverability	347
12.1	Recoverability Theorem	347
12.2	Schatten Norms and Complex Interpolation	348
12.3	Petz Recovery Map	353
12.4	Rényi Information Measure	355
12.5	Proof of the Recoverability Theorem	359
12.6	Refinements of Quantum Entropy Inequalities	362
12.7	History and Further Reading	366
13	The Information of Quantum Channels	370
13.1	Mutual Information of a Classical Channel	372
13.2	Private Information of a Wiretap Channel	379
13.3	Holevo Information of a Quantum Channel	383
13.4	Mutual Information of a Quantum Channel	389
13.5	Coherent Information of a Quantum Channel	393
13.6	Private Information of a Quantum Channel	398
13.7	Summary	404
13.8	History and Further Reading	404

14	Classical Typicality	406
	14.1 An Example of Typicality	407
	14.2 Weak Typicality	408
	14.3 Properties of the Typical Set	410
	14.4 Application: Data Compression	412
	14.5 Weak Joint Typicality	414
	14.6 Weak Conditional Typicality	417
	14.7 Strong Typicality	420
	14.8 Strong Joint Typicality	429
	14.9 Strong Conditional Typicality	431
	14.10 Application: Channel Capacity Theorem	437
	14.11 Concluding Remarks	442
	14.12 History and Further Reading	442
15	Quantum Typicality	443
	15.1 The Typical Subspace	444
	15.2 Conditional Quantum Typicality	454
	15.3 The Method of Types for Quantum Systems	464
	15.4 Concluding Remarks	466
	15.5 History and Further Reading	466
16	The Packing Lemma	467
	16.1 Introductory Example	468
	16.2 The Setting of the Packing Lemma	468
	16.3 Statement of the Packing Lemma	471
	16.4 Proof of the Packing Lemma	472
	16.5 Derandomization and Expurgation	478
	16.6 Sequential Decoding	480
	16.7 History and Further Reading	485
17	The Covering Lemma	486
	17.1 Introductory Example	487
	17.2 Setting and Statement of the Covering Lemma	489
	17.3 Operator Chernoff Bound	491
	17.4 Proof of the Covering Lemma	495
	17.5 History and Further Reading	501
Part V Noiseless Quantum Shannon Theory		503
18	Schumacher Compression	505
	18.1 The Information-Processing Task	506
	18.2 The Quantum Data Compression Theorem	508
	18.3 Quantum Compression Example	512

18.4	Variations on the Schumacher Theme	513
18.5	Concluding Remarks	515
18.6	History and Further Reading	515
19	Entanglement Manipulation	517
19.1	Sketch of Entanglement Manipulation	519
19.2	LOCC and Relative Entropy of Entanglement	522
19.3	Entanglement Manipulation Task	524
19.4	The Entanglement Manipulation Theorem	525
19.5	Concluding Remarks	535
19.6	History and Further Reading	535
	Part VI Noisy Quantum Shannon Theory	537
	Introduction	539
20	Classical Communication	541
20.1	Naive Approach: Product Measurements	543
20.2	The Information-Processing Task	546
20.3	The Classical Capacity Theorem	548
20.4	Examples of Channels	556
20.5	Superadditivity of the Holevo Information	565
20.6	Concluding Remarks	568
20.7	History and Further Reading	569
21	Entanglement-Assisted Classical Communication	571
21.1	The Information-Processing Task	573
21.2	A Preliminary Example	574
21.3	Entanglement-Assisted Capacity Theorem	577
21.4	The Direct Coding Theorem	578
21.5	The Converse Theorem	586
21.6	Examples of Channels	592
21.7	Concluding Remarks	597
21.8	History and Further Reading	598
22	Coherent Communication with Noisy Resources	600
22.1	Entanglement-Assisted Quantum Communication	601
22.2	Quantum Communication	606
22.3	Noisy Super-Dense Coding	607
22.4	State Transfer	610
22.5	Trade-off Coding	614
22.6	Concluding Remarks	622
22.7	History and Further Reading	622

x	Contents	
23	Private Classical Communication	624
23.1	The Information-Processing Task	625
23.2	The Private Classical Capacity Theorem	627
23.3	The Direct Coding Theorem	628
23.4	The Converse Theorem	636
23.5	Discussion of Private Classical Capacity	638
23.6	History and Further Reading	641
24	Quantum Communication	642
24.1	The Information-Processing Task	643
24.2	No-Cloning and Quantum Communication	645
24.3	The Quantum Capacity Theorem	646
24.4	The Direct Coding Theorem	647
24.5	Converse Theorem	654
24.6	An Interlude with Quantum Stabilizer Codes	656
24.7	Example Channels	662
24.8	Discussion of Quantum Capacity	666
24.9	Entanglement Distillation	673
24.10	History and Further Reading	675
25	Trading Resources for Communication	679
25.1	The Information-Processing Task	680
25.2	The Quantum Dynamic Capacity Theorem	682
25.3	The Direct Coding Theorem	687
25.4	The Converse Theorem	688
25.5	Examples of Channels	699
25.6	History and Further Reading	715
26	Summary and Outlook	717
26.1	Unit Protocols	718
26.2	Noiseless Quantum Shannon Theory	718
26.3	Noisy Quantum Shannon Theory	719
26.4	Protocols Not Covered In This Book	722
26.5	Network Quantum Shannon Theory	723
26.6	Future Directions	724
	Appendix A Supplementary Results	726
	Appendix B Unique Linear Extension of a Quantum Physical Evolution	730
	<i>References</i>	733
	<i>Index</i>	754

Preface to the Second Edition

It has now been some years since I completed the first draft of the first edition of this book. In this time, I have learned much from many collaborators and I am grateful to them. During the past few years, Mario Berta, Nilanjana Datta, Saikat Guha, and Andreas Winter have strongly shaped my thinking about quantum information theory, and Mario and Nilanjana in particular have influenced my technical writing style, which is reflected in the new edition of the book. Also, the chance to work with them and others has led me to new research directions in quantum information theory that I never would have imagined on my own.

I am also thankful to Todd Brun, Paul Cuff, Ludovico Lami, Ciara Morgan, and Giannicola Scarpa for using the book as the main text in their graduate courses on quantum information theory and for feedback. One can try as much as possible to avoid typos in a book, but inevitably, they seem to show up in unexpected places. I am grateful to many people for pointing out typos or errors and for suggesting how to fix them, including Todd Brun, Giulio Chiribella, Paul Cuff, Dawei (David) Ding, Will Matthews, Milan Mosonyi, David Reeb, and Marco Tomamichel. I also thank Corsin Pfister for helpful discussions about unique linear extensions of quantum physical evolutions. I am grateful to David Tranah and the editorial staff at Cambridge University Press for their help with publishing the second edition.

So what's new in the second edition? Suffice it to say that every page of the book has been rewritten and there are over 100 pages of new material! I formulated many thoughts about the revision during fall 2013 while teaching a graduate course on quantum information at LSU, and I then formulated many more thoughts and made the actual changes during fall 2015 (when teaching it again). In that regard, I am thankful to both the Department of Physics and Astronomy and the Center for Computation and Technology at LSU for providing a great environment and support. I also thank the graduate students at LSU who gave feedback during and after lectures. There are many little changes throughout that will probably go unnoticed. For example, I have come to prefer writing a quantum state shared between Alice and Bob as ρ_{AB} rather than ρ^{AB} (i.e., with system labels as subscripts rather than superscripts). Admittedly, several collaborators influenced me here, but there are a few good reasons for this convention: the phrase “state of a quantum system” suggests that the

state ρ should be “resting on” the systems AB , the often-used partial trace $\text{Tr}_A\{\rho_{AB}\}$ looks better than $\text{Tr}_A\{\rho^{AB}\}$, and the notation ρ_{AB} is more consistent with the standard notation p_X for a probability distribution corresponding to a random variable X . OK, that’s perhaps minor. Major changes include the addition of many new exercises, a detailed discussion of Bell’s theorem, the CHSH game, and Tsirelson’s theorem, the axiomatic approach to quantum channels, a proof of the Choi–Kraus theorem, a definition of unital and adjoint maps, a discussion of states, channels, and measurements all as quantum channels, the equivalence of purifications, the adjoint map in terms of isometric extension, the definition of the diamond norm and its interpretation, how a measurement achieves the fidelity, how the Hilbert–Schmidt distance is not monotone with respect to channels, more detailed definitions of classical and quantum relative entropies, new continuity bounds for classical and quantum entropies, refinements of classical entropy inequalities, streamlined proofs of data processing inequalities using relative entropy, the equivalence of quantum entropy inequalities like strong subadditivity and monotonicity of relative entropy, Chapter 12 on recoverability, modified proofs of additivity of channel information quantities, sequential decoding for classical communication, simpler proofs of the Schumacher compression theorem, a complete rewrite of Chapter 19, alternate proofs for the achievability part of the HSW theorem, a proof for the classical capacity of the erasure channel, simpler converse proofs for the entanglement-assisted capacity theorem, a revised proof of the trade-off coding resource inequality, a revised proof of the hashing bound, a simplified converse proof of the quantum dynamic capacity theorem, a completely revised discussion of the importance of the quantum dynamic capacity formula, and the addition of many new references that have been influential in recent years. Minor changes include improved presentations of many theorems and definitions throughout.

I am most grateful to my family for all of their support and encouragement throughout my life, including my mother, father, sister, and brother and all of my surrounding family members. I am still indebted to my wife Christabelle and her family for warmth and love. Christabelle has been an unending source of support and love for me. I dedicate this second edition to my nephews David and Matthew.

Preface to the First Edition

I began working on this book in the summer of 2008 in Los Angeles, with much time to spare in the final months of dissertation writing. I had a strong determination to review quantum Shannon theory, a beautiful area of quantum information science that Igor Devetak had taught me three years earlier at USC in fall 2005. I was carefully studying a manuscript entitled “Principles of Quantum Information Theory,” a text that Igor had initiated in collaboration with Patrick Hayden and Andreas Winter. I read this manuscript many times, and many parts of it I understood well, though other parts I did not.

After a few weeks of reading and rereading, I decided “if I can write it out myself from scratch, perhaps I would then understand it!”, and thus began the writing of the chapters on the packing lemma, the covering lemma, and quantum typicality. I knew that Igor’s (now former) students Min-Hsiu Hsieh and Zhicheng Luo knew the topic well because they had already written several quality research papers with him, so I requested if they could meet with me weekly for an hour to review the fundamentals. They kindly agreed and helped me quite a bit in understanding the packing and covering techniques.

After graduating, I began collaborating with Min-Hsiu on a research project that Igor had suggested to the both of us: “find the triple trade-off capacity formulas of a quantum channel.” This was perhaps the best starting point for me to learn quantum Shannon theory because proving this theorem required an understanding of most everything that had already been accomplished in the area. After a month of effort, I continued to work with Min-Hsiu on this project while joining Andreas Winter’s Singapore group for a two-month visit. As I learned more, I added more to the notes, and they continued to grow.

After landing a job in the DC area for January 2009, I realized that I had almost enough material for teaching a course, and so I contacted local universities in the area to see if they would be interested. Can Korman, formerly chair of the Electrical Engineering Department at George Washington University, was excited about the possibility. His enthusiasm was enough to keep me going on the notes, and so I continued to refine and add to them in my spare time in preparing for teaching. Unfortunately (or perhaps fortunately?), the course ended up being canceled. This was disheartening to me, but in the mean time, I had contacted Patrick Hayden to see if he would be interested in having me join his group at

McGill University. Patrick Hayden and David Avis then offered me a postdoctoral fellowship, and I moved to Montréal in October 2009. After joining, I learned a lot by collaborating and discussing with Patrick and his group members. Patrick offered me the opportunity to teach his graduate class on quantum Shannon theory while he was away on sabbatical, and this encouraged me further to persist with the notes.

I am grateful to everyone mentioned above for encouraging and supporting me during this project, and I am also grateful to everyone who provided feedback during the course of writing up. In this regard, I am especially grateful to Dave Touchette for detailed feedback on all of the chapters in the book. Dave's careful reading and spotting of errors has immensely improved the quality of the book. I am grateful to my father, Gregory E. Wilde, Sr., for feedback on earlier chapters and for advice and love throughout. I thank Ivan Savov for encouraging me, for feedback, and for believing that this is an important scholarly work. I also thank Constance Caramanolis, Raza-Ali Kazmi, John M. Schanck, Bilal Shaw, and Anna Vershynina for valuable feedback. I am grateful to Min-Hsiu Hsieh for the many research topics we have worked on together that have enhanced my knowledge of quantum Shannon theory. I thank Michael Nielsen and Victor Shoup for advice on Creative Commons licensing and Kurt Jacobs for advice on book publishing. I am grateful to Sarah Payne and David Tranah of Cambridge University Press for their extensive feedback on the manuscript and their outstanding support throughout the publication process. I acknowledge funding from the MDEIE (Quebec) PSR-SIIRI grant.

I am indebted to my mentors who took me on as a student. Todd Brun was a wonderful PhD supervisor—helpful, friendly, and encouraging of creativity and original pursuit. Igor Devetak taught me quantum Shannon theory in fall 2005 and helped me once per week during his office hours. He also invited me to join Todd's and his group, and more recently, Igor provided much encouragement and “big-picture” feedback during the writing of this book. Bart Kosko shaped me as a scholar during my early years at USC and provided helpful advice regarding the book project. Patrick Hayden has been an immense bedrock of support at McGill. His knowledge of quantum information and many other areas is unsurpassed, and he has been kind, inviting, and helpful during my time at McGill. I am also grateful to Patrick for giving me the opportunity to teach at McGill and for advice throughout the development of this book.

I thank my mother, father, sister, and brother and all of my surrounding family members for being a source of love and support. Finally, I am indebted to my wife Christabelle and her family for warmth and love. I dedicate this book to the memory of my grandparents Joseph and Rose McMahon, and Norbert Jay and Mary Wilde. *Lux aeterna luceat eis, Domine.*

How To Use This Book

For Students

Prerequisites for understanding the content in this book are a solid background in probability theory and linear algebra. If you are new to information theory, then there should be enough background in this book to get you up to speed (Chapters 2, 10, 13, and 14). However, classics on information theory such as Cover & Thomas (2006) and MacKay (2003) could be helpful as a reference. If you are new to quantum mechanics, then there should be enough material in this book (Part II) to give you the background necessary for understanding quantum Shannon theory. The book of Nielsen & Chuang (2000), sometimes affectionately known as “Mike and Ike”, has become the standard starting point for students in quantum information science and might be helpful as well. Some of the content of that book is available in the dissertation of Nielsen (1998). If you are familiar with Shannon’s information theory (at the level of Cover & Thomas, 2006, for example), then the present book should be a helpful entry point into the field of quantum Shannon theory. We build on intuition developed classically to help in establishing schemes for communication over quantum channels. If you are familiar with quantum mechanics, it might still be worthwhile to review Part II because some content there might not be part of a standard course on quantum mechanics.

The aim of this book is to develop “from the ground up” many of the major, exciting pre- and post-millennium developments in the general area of study known as quantum Shannon theory. As such, we spend a significant amount of time on quantum mechanics for quantum information theory (Part II), we give a careful study of the important unit protocols of teleportation, super-dense coding, and entanglement distribution (Part III), and we develop many of the tools necessary for understanding information transmission or compression (Part IV). Parts V and VI are the culmination of this book, where all of the tools developed come into play for understanding many of the important results in quantum Shannon theory.

For Instructors

This book could be useful for self-learning or as a reference, but one of the main goals is for it to be employed as an instructional aid for the classroom.

To aid instructors in designing a course to suit their own needs, a draft, pre-publication copy of this book is available under a Creative Commons Attribution-NonCommercial-ShareAlike license. This means that you can modify and redistribute this draft, pre-publication copy as you wish, as long as you attribute the author, you do not use it for commercial purposes, and you share a modification or derivative work under the same license (for a readable summary of the terms of the license, see <http://creativecommons.org/licenses/by-nc-sa/3.0/>). These requirements can be waived if you obtain permission directly from the author. By releasing the draft, pre-publication copy of the book under this license, I expect and encourage instructors to modify it for their own needs. This will allow for the addition of new exercises, new developments in the theory, and the latest open problems. It might also be a helpful starting point for a book on a related topic, such as network quantum Shannon theory.

I used an earlier version of this book in a one-semester course on quantum Shannon theory at McGill University during the winter semester 2011 (in many parts of the USA, this semester is typically called “spring semester”). We almost went through the entire book, but it might also be possible to spread the content over two semesters instead. Here is the order in which we proceeded:

1. Introduction in Part I.
2. Quantum mechanics in Part II.
3. Unit protocols in Part III.
4. Chapter 9 on distance measures, Chapter 10 on classical information and entropy, and Chapter 11 on quantum information and entropy.
5. The first part of Chapter 14 on classical typicality and Shannon compression.
6. The first part of Chapter 15 on quantum typicality.
7. Chapter 18 on Schumacher compression.
8. Back to Chapters 14 and 15 for the method of types.
9. Chapter 19 on entanglement concentration.
10. Chapter 20 on classical communication.
11. Chapter 21 on entanglement-assisted classical communication.
12. The final explosion of results in Chapter 22 (one of which is a route to proving the achievability part of the quantum capacity theorem).

The above order is just a particular order that suited the needs for the class at McGill, but other orders are of course possible. One could sacrifice the last part of Part III on the unit resource capacity region if there is no desire to cover the quantum dynamic capacity theorem. One could also focus on going from classical communication to private classical communication to quantum communication in order to develop some more intuition behind the quantum capacity theorem. I later did this when teaching the course at LSU in fall 2013. But just recently in fall 2015, I went back to the ordering above while including lectures devoted to the CHSH game and the new results in Chapter 12.

Other Sources

There are many other sources to obtain a background in quantum Shannon theory. The standard reference has become the book of Nielsen & Chuang (2000), but it does not feature any of the post-millennium results in quantum Shannon theory. Other excellent books that cover some aspects of quantum Shannon theory are (Hayashi, 2006, Holevo, 2002a, Holevo, 2012, Watrous, 2015). Patrick Hayden has had a significant hand as a collaborative guide for many PhD and Masters' theses in quantum Shannon theory, during his time as a postdoctoral fellow at the California Institute of Technology and as a professor at McGill University. These include the theses of Yard (2005), Abeyesinghe (2006), Savov (2008), Savov (2012), Dupuis (2010), and Dutil (2011). All of these theses are excellent references. Hayden also had a strong influence over the present author during the development of the first edition of this book.