## Quantum Information Theory

Developing many of the major, exciting pre- and post-millennium developments from the ground up, this book is an ideal entry point for graduate students into quantum information theory. Significant attention is given to quantum mechanics for quantum information theory, and careful studies of the important protocols of teleportation, super-dense coding, and entanglement distribution are presented.

In this new edition, readers can expect to find over 100 pages of new material, including detailed discussions of Bell's theorem, the CHSH game, Tsirelson's theorem, the axiomatic approach to quantum channels, the definition of the diamond norm and its interpretation, and a proof of the Choi–Kraus theorem. Discussion of the importance of the quantum dynamic capacity formula has been completely revised, and many new exercises and references have been added. This new edition will be warmly welcomed by the upcoming generation of quantum information theorists and the already-established community of classical information theorists.

**Mark M. Wilde** is an Assistant Professor in the Department of Physics and Astronomy, and in the Center for Computation and Technology, at Louisiana State University in Baton Rouge, Louisiana. He is the recipient of a National Science Foundation Career Development Award and the APS-IUSSTF Professorship Award in Physics. He is also a Senior Member of the IEEE, and is currently serving as Associate Editor for Quantum Information Theory for the leading journal, *IEEE Transactions on Information Theory*. His current research interests are in quantum Shannon theory, quantum optical communication, quantum computational complexity theory, and quantum error correction.

# Quantum Information Theory

## Second Edition

MARK M. WILDE
Louisiana State University

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107176164



First published 2013
Second edition 2017

Printed in the United Kingdom by Clays, St Ives plc

*A catalogue record for this publication is available from the British Library.*

ISBN 978-1-107-17616-4 Hardback

Cambridge University Press
978-1-107-17616-4 — Quantum Information Theory
Mark M. Wilde
Frontmatter
More Information

# Contents



 

# Preface to the Second Edition

It has now been some years since I completed the first draft of the first edition of this book. In this time, I have learned much from many collaborators and I am grateful to them. During the past few years, Mario Berta, Nilanjana Datta, Saikat Guha, and Andreas Winter have strongly shaped my thinking about quantum information theory, and Mario and Nilanjana in particular have influenced my technical writing style, which is reflected in the new edition of the book. Also, the chance to work with them and others has led me to new research directions in quantum information theory that I never would have imagined on my own.

I am also thankful to Todd Brun, Paul Cuff, Ludovico Lami, Ciara Morgan, and Giannicola Scarpa for using the book as the main text in their graduate courses on quantum information theory and for feedback. One can try as much as possible to avoid typos in a book, but inevitably, they seem to show up in unexpected places. I am grateful to many people for pointing out typos or errors and for suggesting how to fix them, including Todd Brun, Giulio Chiribella, Paul Cuff, Dawei (David) Ding, Will Matthews, Milan Mosonyi, David Reeb, and Marco Tomamichel. I also thank Corsin Pfister for helpful discussions about unique linear extensions of quantum physical evolutions. I am grateful to David Tranah and the editorial staff at Cambridge University Press for their help with publishing the second edition.

So what's new in the second edition? Suffice it to say that every page of the book has been rewritten and there are over 100 pages of new material! I formulated many thoughts about the revision during fall 2013 while teaching a graduate course on quantum information at LSU, and I then formulated many more thoughts and made the actual changes during fall 2015 (when teaching it again). In that regard, I am thankful to both the Department of Physics and Astronomy and the Center for Computation and Technology at LSU for providing a great environment and support. I also thank the graduate students at LSU who gave feedback during and after lectures. There are many little changes throughout that will probably go unnoticed. For example, I have come to prefer writing a quantum state shared between Alice and Bob as $\rho_{AB}$ rather than $\rho^{AB}$ (i.e., with system labels as subscripts rather than superscripts). Admittedly, several collaborators influenced me here, but there are a few good reasons for this convention: the phrase "state of a quantum system" suggests that the state $\rho$ should be "resting on" the systems $AB$, the often-used partial trace $\text{Tr}_A\{\rho_{AB}\}$ looks better than $\text{Tr}_A\{\rho^{AB}\}$, and the notation $\rho_{AB}$ is more consistent with the standard notation $p_X$ for a probability distribution corresponding to a random variable $X$. OK, that's perhaps minor. Major changes include the addition of many new exercises, a detailed discussion of Bell's theorem, the CHSH game, and Tsirelson's theorem, the axiomatic approach to quantum channels, a proof of the Choi–Kraus theorem, a definition of unital and adjoint maps, a discussion of states, channels, and measurements all as quantum channels, the equivalence of purifications, the adjoint map in terms of isometric extension, the definition of the diamond norm and its interpretation, how a measurement achieves the fidelity, how the Hilbert–Schmidt distance is not monotone with respect to channels, more detailed definitions of classical and quantum relative entropies, new continuity bounds for classical and quantum entropies, refinements of classical entropy inequalities, streamlined proofs of data processing inequalities using relative entropy, the equivalence of quantum entropy inequalities like strong subadditivity and monotonicity of relative entropy, Chapter 12 on recoverability, modified proofs of additivity of channel information quantities, sequential decoding for classical communication, simpler proofs of the Schumacher compression theorem, a complete rewrite of Chapter 19, alternate proofs for the achievability part of the HSW theorem, a proof for the classical capacity of the erasure channel, simpler converse proofs for the entanglement-assisted capacity theorem, a revised proof of the trade-off coding resource inequality, a revised proof of the hashing bound, a simplified converse proof of the quantum dynamic capacity theorem, a completely revised discussion of the importance of the quantum dynamic capacity formula, and the addition of many new references that have been influential in recent years. Minor changes include improved presentations of many theorems and definitions throughout.

I am most grateful to my family for all of their support and encouragement throughout my life, including my mother, father, sister, and brother and all of my surrounding family members. I am still indebted to my wife Christabelle and her family for warmth and love. Christabelle has been an unending source of support and love for me. I dedicate this second edition to my nephews David and Matthew.

# Preface to the First Edition

I began working on this book in the summer of 2008 in Los Angeles, with much time to spare in the final months of dissertation writing. I had a strong determination to review quantum Shannon theory, a beautiful area of quantum information science that Igor Devetak had taught me three years earlier at USC in fall 2005. I was carefully studying a manuscript entitled "Principles of Quantum Information Theory," a text that Igor had initiated in collaboration with Patrick Hayden and Andreas Winter. I read this manuscript many times, and many parts of it I understood well, though other parts I did not.

After a few weeks of reading and rereading, I decided "if I can write it out myself from scratch, perhaps I would then understand it!", and thus began the writing of the chapters on the packing lemma, the covering lemma, and quantum typicality. I knew that Igor's (now former) students Min-Hsiu Hsieh and Zhicheng Luo knew the topic well because they had already written several quality research papers with him, so I requested if they could meet with me weekly for an hour to review the fundamentals. They kindly agreed and helped me quite a bit in understanding the packing and covering techniques.

After graduating, I began collaborating with Min-Hsiu on a research project that Igor had suggested to the both of us: "find the triple trade-off capacity formulas of a quantum channel." This was perhaps the best starting point for me to learn quantum Shannon theory because proving this theorem required an understanding of most everything that had already been accomplished in the area. After a month of effort, I continued to work with Min-Hsiu on this project while joining Andreas Winter's Singapore group for a two-month visit. As I learned more, I added more to the notes, and they continued to grow.

After landing a job in the DC area for January 2009, I realized that I had almost enough material for teaching a course, and so I contacted local universities in the area to see if they would be interested. Can Korman, formerly chair of the Electrical Engineering Department at George Washington University, was excited about the possibility. His enthusiasm was enough to keep me going on the notes, and so I continued to refine and add to them in my spare time in preparing for teaching. Unfortunately (or perhaps fortunately?), the course ended up being canceled. This was disheartening to me, but in the mean time, I had contacted Patrick Hayden to see if he would be interested in having me join his group at

McGill University. Patrick Hayden and David Avis then offered me a postdoctoral fellowship, and I moved to Montréal in October 2009. After joining, I learned a lot by collaborating and discussing with Patrick and his group members. Patrick offered me the opportunity to teach his graduate class on quantum Shannon theory while he was away on sabbatical, and this encouraged me further to persist with the notes.

I am grateful to everyone mentioned above for encouraging and supporting me during this project, and I am also grateful to everyone who provided feedback during the course of writing up. In this regard, I am especially grateful to Dave Touchette for detailed feedback on all of the chapters in the book. Dave's careful reading and spotting of errors has immensely improved the quality of the book. I am grateful to my father, Gregory E. Wilde, Sr., for feedback on earlier chapters and for advice and love throughout. I thank Ivan Savov for encouraging me, for feedback, and for believing that this is an important scholarly work. I also thank Constance Caramanolis, Raza-Ali Kazmi, John M. Schanck, Bilal Shaw, and Anna Vershynina for valuable feedback. I am grateful to Min-Hsiu Hsieh for the many research topics we have worked on together that have enhanced my knowledge of quantum Shannon theory. I thank Michael Nielsen and Victor Shoup for advice on Creative Commons licensing and Kurt Jacobs for advice on book publishing. I am grateful to Sarah Payne and David Tranah of Cambridge University Press for their extensive feedback on the manuscript and their outstanding support throughout the publication process. I acknowledge funding from the MDEIE (Quebec) PSR-SIIRI grant.

I am indebted to my mentors who took me on as a student. Todd Brun was a wonderful PhD supervisor—helpful, friendly, and encouraging of creativity and original pursuit. Igor Devetak taught me quantum Shannon theory in fall 2005 and helped me once per week during his office hours. He also invited me to join Todd's and his group, and more recently, Igor provided much encouragement and "big-picture" feedback during the writing of this book. Bart Kosko shaped me as a scholar during my early years at USC and provided helpful advice regarding the book project. Patrick Hayden has been an immense bedrock of support at McGill. His knowledge of quantum information and many other areas is unsurpassed, and he has been kind, inviting, and helpful during my time at McGill. I am also grateful to Patrick for giving me the opportunity to teach at McGill and for advice throughout the development of this book.

I thank my mother, father, sister, and brother and all of my surrounding family members for being a source of love and support. Finally, I am indebted to my wife Christabelle and her family for warmth and love. I dedicate this book to the memory of my grandparents Joseph and Rose McMahon, and Norbert Jay and Mary Wilde. *Lux aeterna luceat eis, Domine.*

# How To Use This Book

## For Students

Prerequisites for understanding the content in this book are a solid background in probability theory and linear algebra. If you are new to information theory, then there should be enough background in this book to get you up to speed (Chapters 2, 10, 13, and 14). However, classics on information theory such as Cover & Thomas (2006) and MacKay (2003) could be helpful as a reference. If you are new to quantum mechanics, then there should be enough material in this book (Part II) to give you the background necessary for understanding quantum Shannon theory. The book of Nielsen & Chuang (2000), sometimes affectionately known as "Mike and Ike", has become the standard starting point for students in quantum information science and might be helpful as well. Some of the content of that book is available in the dissertation of Nielsen (1998). If you are familiar with Shannon's information theory (at the level of Cover & Thomas, 2006, for example), then the present book should be a helpful entry point into the field of quantum Shannon theory. We build on intuition developed classically to help in establishing schemes for communication over quantum channels. If you are familiar with quantum mechanics, it might still be worthwhile to review Part II because some content there might not be part of a standard course on quantum mechanics.

The aim of this book is to develop "from the ground up" many of the major, exciting pre- and post-millennium developments in the general area of study known as quantum Shannon theory. As such, we spend a significant amount of time on quantum mechanics for quantum information theory (Part II), we give a careful study of the important unit protocols of teleportation, super-dense coding, and entanglement distribution (Part III), and we develop many of the tools necessary for understanding information transmission or compression (Part IV). Parts V and VI are the culmination of this book, where all of the tools developed come into play for understanding many of the important results in quantum Shannon theory.

## For Instructors

This book could be useful for self-learning or as a reference, but one of the main goals is for it to be employed as an instructional aid for the classroom.

 

To aid instructors in designing a course to suit their own needs, a draft, pre-publication copy of this book is available under a Creative Commons Attribution-NonCommercial-ShareAlike license. This means that you can modify and redistribute this draft, pre-publication copy as you wish, as long as you attribute the author, you do not use it for commercial purposes, and you share a modification or derivative work under the same license (for a readable summary of the terms of the license, see `http://creativecommons.org/licenses/by-nc-sa/3.0/`). These requirements can be waived if you obtain permission directly from the author. By releasing the draft, pre-publication copy of the book under this license, I expect and encourage instructors to modify it for their own needs. This will allow for the addition of new exercises, new developments in the theory, and the latest open problems. It might also be a helpful starting point for a book on a related topic, such as network quantum Shannon theory.

I used an earlier version of this book in a one-semester course on quantum Shannon theory at McGill University during the winter semester 2011 (in many parts of the USA, this semester is typically called "spring semester"). We almost went through the entire book, but it might also be possible to spread the content over two semesters instead. Here is the order in which we proceeded:

1. Introduction in Part I.
2. Quantum mechanics in Part II.
3. Unit protocols in Part III.
4. Chapter 9 on distance measures, Chapter 10 on classical information and entropy, and Chapter 11 on quantum information and entropy.
5. The first part of Chapter 14 on classical typicality and Shannon compression.
6. The first part of Chapter 15 on quantum typicality.
7. Chapter 18 on Schumacher compression.
8. Back to Chapters 14 and 15 for the method of types.
9. Chapter 19 on entanglement concentration.
10. Chapter 20 on classical communication.
11. Chapter 21 on entanglement-assisted classical communication.
12. The final explosion of results in Chapter 22 (one of which is a route to proving the achievability part of the quantum capacity theorem).

The above order is just a particular order that suited the needs for the class at McGill, but other orders are of course possible. One could sacrifice the last part of Part III on the unit resource capacity region if there is no desire to cover the quantum dynamic capacity theorem. One could also focus on going from classical communication to private classical communication to quantum communication in order to develop some more intuition behind the quantum capacity theorem. I later did this when teaching the course at LSU in fall 2013. But just recently in fall 2015, I went back to the ordering above while including lectures devoted to the CHSH game and the new results in Chapter 12.

## Other Sources

There are many other sources to obtain a background in quantum Shannon theory. The standard reference has become the book of Nielsen & Chuang (2000), but it does not feature any of the post-millennium results in quantum Shannon theory. Other excellent books that cover some aspects of quantum Shannon theory are (Hayashi, 2006, Holevo, 2002a, Holevo, 2012, Watrous, 2015). Patrick Hayden has had a significant hand as a collaborative guide for many PhD and Masters' theses in quantum Shannon theory, during his time as a postdoctoral fellow at the California Institute of Technology and as a professor at McGill University. These include the theses of Yard (2005), Abeyesinghe (2006), Savov (2008), Savov (2012), Dupuis (2010), and Dutil (2011). All of these theses are excellent references. Hayden also had a strong influence over the present author during the development of the first edition of this book.