LDPC CODE DESIGNS, CONSTRUCTIONS, AND UNIFICATION

Written by leading experts, this self-contained text provides systematic coverage of low-density parity-check (LDPC) codes and their construction techniques, unifying both algebraic- and graph-based approaches into a single theoretical framework (the superposition construction). An algebraic method for constructing protograph LDPC codes is described, and entirely new codes and techniques are presented. These include a new class of LDPC codes with doubly quasi-cyclic structure, as well as algebraic methods for constructing spatially and globally coupled LDPC codes.

Authoritative, yet written using accessible language, this is essential reading for electrical engineers, computer scientists, and mathematicians working in communications and information theory.

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“This book provides an in-depth survey of recently developed quasi-cyclic LDPC codes. It is a treasured reference on practical channel coding methods for both theorists and practitioners working in communications and information theory.”

Lara Dolecek, University of California, Los Angeles

“The importance of LDPC codes in numerous applications and their capacity-approaching performance has led to an explosion in research into their construction and analysis over the past decade. The numerous effective constructions of them can be broadly classified as algebraic and graphical, including the important superposition, protograph, and spatial coupling techniques. This timely volume explains, unifies, and greatly clarifies these diverse approaches and lays a solid foundation that will be invaluable to researchers, practitioners, and students alike.”

Ian F. Blake, University of British Columbia

“A book from the leaders in the field of error-correcting codes. Superposition – a unified framework for low-density parity check code construction – makes a description of codes of various classes rather simple.”

Bane Vasic, University of Arizona, Tucson
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Preface

Error control codes protect the accuracy of data in modern information systems, including computing, communication, and storage systems. Low-density parity-check (LDPC) codes and their relatives represent the state of the art in error control coding and are renowned for their ability to perform close to the theoretical limits. This book presents recent results on various LDPC code designs, making strong connections between two prominent design approaches, the algebraic-based and the graph-theoretic-based constructions. New codes and code construction techniques are presented.

Most methods for constructing LDPC codes can be classified into two general categories, the algebraic-based and the graph-theoretic-based constructions. The two best-known graph-theoretic-based construction methods are the progressive edge-growth (PEG) and the protograph-based (PTG-based) methods, devised in 2001 and 2003, respectively. Both of these techniques involve computer-aided design. One of the earliest algebraic-based methods for constructing LDPC codes is the superposition (SP) construction, proposed in 2002. In this book, the algebraic-based construction method is re-interpreted from both the algebraic and the graph-theoretic perspectives. From the algebraic point of view, it is shown that the SP-construction of LDPC codes includes, as special cases, most of the major algebraic construction methods developed since 2002. From the graph-theoretic point of view, it is shown that the SP-construction also includes the PTG-based construction as a special case. Based on this PTG/SP connection, an algebraic method is developed here to construct PTG-based LDPC codes.

There are advantages to putting the algebraic-based and the PTG-based constructions into a single framework, the SP framework. One advantage is that SP descriptions of codes tend to be relatively compact, enabling simple code specifications in standards and textbooks. Another advantage to studying LDPC codes under the SP framework is that students and practitioners need only learn
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a single code design approach rather than the myriad approaches that exist in the published literature.

Both binary and nonbinary code constructions will be presented under the SP framework. The SP-construction also leads to a new class of LDPC codes with a doubly quasi-cyclic (QC) structure as well as algebraic methods for constructing spatially and globally coupled LDPC codes. The globally coupled codes will be shown to possess a highly effective burst-erasure correction capability.

A good number of new LDPC codes are constructed and simulated over the binary-input additive white Gaussian noise channel and the binary erasure channel.

This book will open the door for readers to understand many topics in modern LDPC codes that are scattered in the literature. It is intended as a self-study guide for students, researchers, and engineers interested in LDPC codes and their variations. The book explains the different design methodologies in detail and provides an ample number of code constructions along with simulations. The book shows that code design and construction are more of an art rather than science. Hopefully, after reading this book, the reader may gain enough artistic experience to produce codes that not only meet required specifications but also improve upon those reported here. To make the material widely accessible, the authors have kept the presentation as clear as possible and assumed only basic knowledge of terminology and results that are commonly covered in textbooks on coding theory.

Acknowledgment

This book is the climax of more than fifteen years of work on LDPC codes by graduate students of the last four authors. In particular, the second and third authors would like to thank Drs. Lei Chen, Qiuju Diao, Ivana Djurdjevic, Qin Huang, Jingyu Kang, Yu Kou, Lan Lan, Keke Liu, Shumei Song, Ying Yu Tai, Heng Tang, Jun Xu, Lingqi Zeng, Li Zhang, and Bo Zhou. Each of them spent thousands of hours investigating choices of code parameters in order to optimize performance. We are also grateful to the federal agencies NASA and NSF and to the industrial support from Cadence, Denali Software, Intel, LSI, Northrop Grumman Space Technology, and San Disk. Many thanks to Dr. Julie Lancashire at Cambridge University Press for her constant encouragement to write this book. We also acknowledge Ms. Isabella Yeh, whose painting on the front cover was created specifically for this book. Finally, the first author would like to thank her father and mother for their love, and the last four authors would like to thank their wives for their patience and understanding. To the six of them, this work is dedicated.