Representation theory of big groups is an important and quickly developing part of modern mathematics, giving rise to a variety of important applications in probability and mathematical physics. This book provides the first concise and self-contained introduction to the theory on the simplest, yet very nontrivial, example of the infinite symmetric group, focusing on its deep connections to probability, mathematical physics, and algebraic combinatorics.

Following a discussion about the classical Thoma theorem, which describes the characters of the infinite symmetric group, the authors describe explicit constructions of an important class of representations, including both the irreducible and generalized regular ones. Complete with detailed proofs, as well as numerous examples and exercises which help to summarize recent developments in the field, this book will enable graduates to enhance their understanding of the topic while also aiding lecturers and researchers in related areas.

Alexei Borodin is a Professor of Mathematics at the Massachusetts Institute of Technology. He also holds the position of Principal Researcher at the Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow.

Grigori Olshanski is a Principal Researcher in the Section of Algebra and Number Theory at the Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow. He also holds the position of Dobrushin Professor at the National Research University Higher School of Economics, Moscow.
Representations of the Infinite Symmetric Group

ALEXEI BORODIN
Massachusetts Institute of Technology
and
Institute for Information Transmission Problems of the Russian Academy of Sciences

GRIGORI OLSHANSKI
Institute for Information Transmission Problems of the Russian Academy of Sciences
and
National Research University Higher School of Economics, Moscow
# Contents

**Introduction**  
**page 1**

**PART ONE  SYMMETRIC FUNCTIONS AND THOMA’S THEOREM**

1 Preliminary Facts From Representation Theory of Finite Symmetric Groups  
1.1 Exercises  
1.2 Notes  
**21**

2 Theory of Symmetric Functions  
2.1 Exercises  
2.2 Notes  
**31**

3 Coherent Systems on the Young Graph  
3.1 The Infinite Symmetric Group and the Young Graph  
3.2 Coherent Systems  
3.3 The Thoma Simplex  
3.4 Integral Representation of Coherent Systems and Characters  
3.5 Exercises  
3.6 Notes  
**37**

4 Extreme Characters and Thoma’s Theorem  
4.1 Thoma’s Theorem  
4.2 Multiplicativity  
4.3 Exercises  
4.4 Notes  
**47**

5 A Toy Model (the Pascal Graph) and de Finetti’s Theorem  
5.1 Exercises  
5.2 Notes  
**55**
6 Asymptotics of Relative Dimension in the Young Graph 62
6.1 Relative Dimension and Shifted Schur Polynomials 62
6.2 The Algebra of Shifted Symmetric Functions 65
6.3 Modified Frobenius Coordinates 66
6.4 The Embedding $\gamma_n \rightarrow \Omega$ and Asymptotic Bounds 68
6.5 Integral Representation of Coherent Systems: Proof 71
6.6 The Vershik–Kerov Theorem 74
6.7 Exercises 75
6.8 Notes 80

7 Boundaries and Gibbs Measures on Paths 82
7.1 The Category $\mathcal{B}$ 82
7.2 Projective Chains 84
7.3 Graded Graphs 86
7.4 Gibbs Measures 88
7.5 Examples of Path Spaces for Branching Graphs 90
7.6 The Martin Boundary and the Vershik–Kerov Ergodic Theorem 91
7.7 Exercises 93
7.8 Notes 96

PART TWO UNITARY REPRESENTATIONS

8 Preliminaries and Gelfand Pairs 101
8.1 Exercises 110
8.2 Notes 113

9 Classification of General Spherical Type Representations 114
9.1 Notes 120

10 Realization of Irreducible Spherical Representations of $(S(\infty) \times S(\infty), \text{diag}S(\infty))$ 121
10.1 Exercises 126
10.2 Notes 128

11 Generalized Regular Representations $T_z$ 130
11.1 Exercises 139
11.2 Notes 140

12 Disjointness of Representations $T_z$ 141
12.1 Preliminaries 141
12.2 Reduction to Gibbs Measures 143
12.3 Exclusion of Degenerate Paths 144
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.4 Proof of Disjointness</td>
<td>146</td>
</tr>
<tr>
<td>12.5 Exercises</td>
<td>149</td>
</tr>
<tr>
<td>12.6 Notes</td>
<td>149</td>
</tr>
</tbody>
</table>

**References**  
150

**Index**  
158