Mathematical Aspects of Signal Processing

Signal processing requires application of techniques and tools to extract information from a given set of data, or conversely, to generate data with some desirable property. For any comprehensive understanding of signal processing, a study of mathematical methods is beneficial because signal processing is associated in significant ways with the interpretation, interrelation and applicability of essential mathematical concepts. The intricacies of signal processing get appreciated better if one understands the relationship between mathematical concepts and signal processing applications.

This text offers detailed discussion on mathematical concepts and their interpretations in the field of signal processing. The purpose of the book is to demonstrate how advanced signal processing applications are structured on basic mathematical concepts. It identifies four areas of mathematics including function representation, generalized inverse, modal decomposition and optimization for detailed treatment. It illustrates how basic concepts transcend into mathematical methods, and mathematical methods perform signal processing tasks. While presenting theorems of mathematics, the emphasis is on applications and interpretations. In order to introduce the computational side of signal processing algorithms, MATLAB programs are referred for simulation, and graphical representations of the simulation results are provided with the concepts wherever required. A number of signal processing applications and practice problems are also embedded in the text, for better understanding of the concepts.

The book targets senior undergraduate and graduate students from multiple disciplines of science and engineering, interested in computations and data processing. Researchers and practicing engineers will be equally benefited by its use as reference.

Pradip Sircar is a Professor of Electrical Engineering at the Indian Institute of Technology, Kanpur. During his teaching career, spanning over three decades, he has guided more than 50 master’s and doctoral theses. He has served on the editorial board of several journals, and has contributed to the quality assessment of higher education in India.

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Dedicated to the memory of my parents

Amiya (Putul) and Nalini Ranjan Sircar

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Foreword

Signal processing is a fundamental discipline and plays a key role in many fields including astronomy, biology, chemistry, economics, engineering, and physics, etc. Signal processing is based on fundamental mathematical tools and methods. As a result, a deeper understanding of these mathematical aspects is crucial towards successful application of signal processing in practice and to carry out advanced research. The aim of the book *Mathematical Aspects of Signal Processing* by Professor Pradip Sircar is to provide the reader with a comprehensive review, along with typical applications, of four specific mathematical topics: Function representation, generalized inverse, modal decomposition, and optimization in signal processing. This book is based on the extensive experience of the author as a researcher and educator in signal processing, and the presented materials have been tested with many students over a number of years at the Indian Institute of Technology, Kanpur, India. In my opinion, this book will be a very good text for an upper division and/or graduate level course on the subject. Researchers and professional engineers will also find this book very useful as a reference.

Sanjit K. Mitra
Santa Barbara, California
January 2016

Preface

I have been using the material of this book for more than two decades while teaching a foundation level course on mathematical methods in signal processing being offered to the final-year undergraduate and first-year postgraduate students of Electrical Engineering at Indian Institute of Technology (IIT) Kanpur. Signal processing being an application area of mathematics, in order to motivate students to get involved in research of signal processing, it was felt that mathematics has to be presented in an informal and thought-provoking manner. In fact, the application area of signal processing is primarily concerned with the interpretation, interrelation, and applicability of mathematical concepts, whereas a formal course of mathematics is structured on treatise of existence, uniqueness, and convergence of mathematical results. Therefore, one has to relook at mathematics and find ways to make mathematical concepts go beyond abstractions and formal proofs.

My experimentation in teaching the above mentioned course has been quite rewarding. I have identified four areas of mathematics, namely, function representation, generalized inverse, modal decomposition, and optimization for our discussions. Mathematical concepts are frequently borrowed from these four areas for signal processing applications. However, the list of areas is not complete. Fundamental results of each area are presented in the class, followed by an interactive discussion of what can be their plausible implications. Finally, signal processing applications are showcased as illustrations of basic principles. Thus, the course serves the purpose of demonstrating how advanced signal processing applications are structured on basic mathematical concepts. I know a good number of students who credited the course, and subsequently decided to work on a research problem related to signal processing applications. The book will motivate students to develop a career in the broad area of signal processing as teachers, researchers or professionals.

During the year 1998–99, while I was visiting École Nationale Supérieure des Télécommunications (ENST) Paris, I started framing the idea of writing a book on the course material. I realized soon that it will not be easy to write a book on this subject. My lectures are more like discussion sessions, and the comprehensions are informal. In a few months time, I jotted down an outline of the proposed course, and drafted some sections of a chapter. However, I could not continue my endeavour after coming back to Kanpur because of other priorities. Nevertheless, the course material was constantly updated over the years.

While selecting the chapters of the book, it was felt that a course on statistical signal processing which requires a background in linear algebra, probability and random processes is offered in the majority of postgraduate programs around the world. Similarly, a foundation course on Fourier, Gabor and wavelet transforms and their applications in signal processing is available in the curriculum at most departments in various universities. However, the scopes of function representation and generalized inverse in signal processing applications are yet to be thoroughly revealed. In the same way, the uses of eigenfunction expansion and mathematical programming in signal processing applications are yet to be fully exploited. These factors were considered while formalizing the contents of this book.

In 2010, I visited the Center for Remote Sensing of Ice Sheets (CReSIS), a research center funded by National Science Foundation (NSF). The center is attached to the University of Kansas at Lawrence, Kansas. The researchers of the center are engaged in some challenging signal processing tasks with real data collected from the planet's polar regions. While participating in various discussion sessions here, I felt the need of having a reference which will show the genesis of many sophisticated signal processing algorithms. It was during this time that I realized again that I should complete writing this book as soon as possible. The bridge between mathematics and signal processing should always be in sight when we are dealing with sophisticated signal processing techniques.

Finally, it should be emphasized that the proposed book does not intend to replace standard texts on mathematics. The purpose of the book is to demonstrate how advanced signal processing applications are structured on basic mathematical concepts. Once students or researchers see the connection, they really get motivated for more exploration. The experiment was very successful at IIT Kanpur. It should be pointed out that the approach taken in this book is the forward approach where we first learn concepts from mathematics, and then look for applications in signal processing. It is felt that the reverse approach of finding mathematical tools for solving signal processing problems does not exploit mathematics to its true potential. It is likely that the book will provide a very distinct flavor.

Acknowledgments

Writing a book is a great learning experience. While learning and framing various underlying concepts leading to different fields of knowledge, one realizes how much one must have acquired from one's teachers. It is natural to have a deep sense of gratitude to all our teachers whose contributions are invaluable. I express my gratitude to Professor Tapan K. Sarkar for all the knowledge that I have acquired from him.

I am grateful to Professor Sanjit K. Mitra for all the encouragements that I received from him every now and then. His enthusiasm and appreciation never failed to boost my spirit. I am indebted to Professor Yves Greiner for all his support in my professional career. He always stood by my side as a true friend. I am very thankful to Professor Sivaprasad Gogineni who entrusted me with a big project which provided ample opportunity for learning.

While teaching the course with the same material at IIT Kanpur, I often got various requests and suggestions from my students. Their critical assessment of the course was extremely helpful in improving the quality of the course material. I sincerely acknowledge this contribution from my students in designing the content of the book. I am very thankful to all my colleagues of our institute for providing a congenial environment where ideas can grow with cherished academic freedom.

I greatly appreciate the kind cooperation that I received from Manish Choudhary of Cambridge University Press from inception to completion of this project. Likewise I thank the editorial team at CUP for all the good work that they have done.