THEORETICAL MANTLE DYNAMICS

Geodynamics is the study of the deformation and flow of the solid Earth and other planetary interiors. Focusing on the Earth's mantle, this book provides a comprehensive, mathematically advanced treatment of the continuum mechanics of mantle processes and the craft of formulating geodynamic models to approximate them. Topics covered include slow viscous flow, elasticity and viscoelasticity, boundary-layer theory, long-wave theories including lubrication theory and shell theory, two-phase flow, and hydrodynamic stability and thermal convection. A unifying theme is the utility of powerful general methods (dimensional analysis, scaling analysis, and asymptotic analysis) that can be applied in many specific contexts. Featuring abundant exercises with worked solutions for graduate students and researchers, this book will make a useful resource for Earth scientists and applied mathematicians with an interest in mantle dynamics and geodynamics more broadly.

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To Anne, Isabelle and Aline, *avec toute ma tendresse*

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Preface

Mantle dynamics, as the term is used in this book, is the study of how the Earth's mantle deforms and flows over long ($\geq 10^2$–10^3 years) time scales. The fundamental concept underlying the theory of mantle dynamics is the notion of a continuous medium. In reality, the rocks of the Earth's mantle are aggregates of discrete mineral grains with characteristic sizes on the order of millimetres. By contrast, the typical length scales of the deformation associated with mantle convection are on the order of 10–10000 km, some 10^7–10^{10} times larger than the grain size. This means that convective motions are not directly influenced by the grain-scale structure of the mantle, but only by the average properties of material elements comprising very large numbers of grains. From a dynamic point of view, therefore, the mantle can be regarded as a continuous medium characterized by physical properties such as density and viscosity that vary smoothly as functions of position. The branch of mechanics that deals with media of this kind is called continuum mechanics.

The basis of continuum mechanics is a set of general conservation laws for mass, momentum and energy, which are usually formulated as partial differential equations. However, because those equations govern all deformations and flows, they describe none in particular and must therefore be supplemented by material constitutive relations and boundary and initial conditions that are appropriate for a particular phenomenon of interest. The result, often called a model problem, is the ultimate object of study in continuum mechanics.

Once posed, a model problem can be solved in one or more of three ways. The first is to construct a physical analog in the laboratory and let nature do the solving. The experimental approach has long played a central role in mantle dynamics, enabling the discovery of many hitherto unexpected phenomena (Davaille and Limare, 2015). The second possibility is to solve the model problem numerically on a computer, which is the dominant approach in present-day mantle dynamics (Ismail-Zadeh and Tackley, 2010). The third approach, the subject of this book, is to solve the problem analytically.

Analytical approaches are, of course, most effective when the model problem at hand is relatively simple and lack some of the flexibility of the best experimental and numerical methods. However, they compensate for this by providing a degree of understanding and insight that no other method can match. Furthermore, an analytical approach is often a necessary preparatory step to subsequent numerical work. Examples treated in this book include Green functions and propagator matrices for Stokes flow driven by internal loads in a spherical annulus (§ 4.9.1, 4.9.2), the governing equations for two-phase flow (§ 9) and the anelastic liquid equations (§ 11.1). Finally, analytical methods play a critical role in the interpretation of experimental and numerical results. For example, dimensional analysis is required to ensure proper scaling of experimental and numerical results to the Earth, and local scaling analysis applied to numerical output can reveal underlying laws that are obscured by numerical tables and graphical images. For all these reasons, the central role that analytical approaches have always played in mantle dynamics is unlikely to diminish.

This book is intended as a combined monograph and textbook for advanced graduate students and researchers working in the field of mantle dynamics. Its seed was a chapter entitled 'Analytical Approaches to Mantle Dynamics' in *Treatise on Geophysics* (Ribe, 2015). The present book more than doubles the length of that chapter by incorporating four types of new material. First, there are expanded derivations and explanations of many topics and results that were merely mentioned in the chapter for lack of space. Examples include the reversibility of Stokes flow, dissipation theorems, non-Newtonian corner flow, viscous eddies, the Stokeslet, slender-body theory, the lubrication theory equations, plume–plate interaction models, effective boundary conditions from thin-layer flows, conduit solitary waves, the thin viscous-sheet equations in two dimensions, matching conditions at a fluid–fluid interface, convection with temperature-dependent viscosity, convection in a compositionally layered mantle and convection with a phase transition. Second, many entirely new topics have been added, including the singularity method for Stokes flow, surface loading of an elastic lithosphere, plumes from a point source of buoyancy in fluids with both constant and temperature-dependent viscosity, the theory of thin viscous sheets in general coordinates, the theory of two-phase flow and compressible convection and the anelastic liquid approximation. Third, nearly 40 new explanatory figures have been included, more than tripling the number that appeared in the *Treatise* chapter. Finally, the book includes some 50 exercises at various levels of difficulty, all with completely worked-out solutions.

Given the advanced nature of the book, a certain amount of preliminary knowledge will be helpful. On the physical side, the main prerequisite is familiarity with basic concepts of general fluid dynamics and low Reynolds number flow at the level of § 2.1–2.3, 3.1–3.4, 3.6–3.7, 4.1–4.2 and 4.7–4.9 of Batchelor (1967), and with the elementary theory of elasticity at the level of chapter 1 of Landau and Lifshitz (1986). Familiarity with geodynamical applications of continuum mechanics (e.g., Turcotte and Schubert, 2014) is welcome but not obligatory. On the mathematical side, the level of the book is admittedly high, but no higher than the best work in the field requires. Frequently used concepts and techniques include Cartesian tensors, linear algebra and the theory of matrices, vector calculus, ordinary and partial differential equations, Fourier series and the Fourier transform and (to a lesser extent) integral equations and the theory of complex variables. More recondite topics such as the differential geometry of surfaces and covariant continuum mechanics also make an appearance, but the necessary elements are reviewed in some detail before they are needed.

However, even if you are not familiar with some of the topics just listed, be assured that I have endeavoured to make the book as widely accessible as possible. I have done this, first, by starting each derivation from first principles, without presupposing that you already know them. Second, I have made an effort to ensure that all derivations are 'followable' from the first step to the last, even if some of the intermediate algebraic steps are only described instead of being written out explicitly. The third means is the many exercises with worked-out solutions, which are specifically designed to help you master the mathematical methods required by the material. Certain of the exercises will be easier if you use a symbolic manipulation package, such as Mathematica© or Maple©, and are indicated by the character string [SM]. You will also be able to construct your own additional exercises by filling in intermediate algebraic steps in the various derivations presented.

Theoretical approaches to mantle dynamics are quite diverse and call for a correspondingly broad and comprehensive treatment. However, it is equally important to highlight the common structures and styles of argumentation that give theoretical mantle dynamics its unity. In this spirit, I begin with a discussion of the craft of formulating geodynamic model problems, focusing on three paradigmatic phenomena (heat transfer from magma diapirs, subduction and plume–lithosphere interaction) that will subsequently reappear in the course of the book treated by different methods. Thus heat transfer from diapirs is treated using dimensional analysis (§ 2.1 and 2.2), scaling analysis (§ 2.3, § 6.5.2) and boundary-layer theory (§ 6.2.2); subduction using dimensional analysis (§ 2.1), scaling analysis based on thin-sheet theory (§ 8.4.1) and the boundary-integral approach (§ 8.4.2); and plume–lithosphere interaction using lubrication theory and scaling analysis (§ 7.1, 7.2). Moreover, the discussions of these and other phenomena are organized as much as possible around three recurrent themes. The first is the importance of scaling arguments (and the scaling laws to which they lead) as tools for understanding physical mechanisms and applying model results to the Earth. Examples of scaling arguments can be found in § 2.3, 3.4, 6.3, 6.4, 6.5.1, 6.5.2, 7.1, 7.2, 8.1.3, 8.4.1, 10.3.3, 10.3.4, 10.4.1, 10.4.4, 10.4.5, 11.1 and 11.2. The second is the ubiquity of self-similar behaviour in geophysical flows, which typically occurs in parts of the spatiotemporal model domain that are sufficiently far from the inhomogeneous initial or boundary conditions that drive the flow to be uninfluenced by the details of those conditions (§ 3.1, 3.2, 3.3, 4.3.3, 4.3.4, 6.2.1, 6.5, 6.5.1, 7.1, 7.2, 10.4.4). The third theme is asymptotic analysis, in which the smallness or largeness of some key parameter in the model problem is exploited to simplify the governing equations, often via a reduction of their dimensionality (3.1, 4.8, 6.1, 6.3, 6.4, 6.5.1, 7.1, 7.3, 7.4, 7.5, 8, 10.3.3, 10.3.4, 10.4.2, 10.4.3, 10.4.4). While these themes by no means encompass everything the book contains, they can serve as threads to guide the reader through what might otherwise appear a trackless labyrinth of miscellaneous methods.

A final aim of the book is to introduce some less-familiar methods that deserve to be better known among geodynamicists. Examples include the use of Papkovich–Fadle eigenfunction expansions (§ 4.4.1) and complex variables (§ 4.5) for 2-D Stokes flows, solutions of the Stokes equations in bispherical coordinates (§ 4.4.3), thin-sheet theory in general nonorthogonal coordinates (§ 8.1) and multiple-scale analysis for modulated convection rolls (§ 10.3).

Throughout this book, unless otherwise stated, Greek indices range over the values 1 and 2; Latin indices range over 1, 2 and 3; and the standard summation convention over repeated subscripts is assumed. Index notation (e.g., u_i, σ_{ij}) and coordinate-free notation ($\mathbf{u}$, $\boldsymbol{\sigma}$) are used interchangeably as convenience dictates. For simplicity, both u_i and $\mathbf{u}$ are called vectors and both σ_{ij} and $\boldsymbol{\sigma}$ tensors, although strictly speaking u_i and σ_{ij} are the components of $\mathbf{u}$ and $\boldsymbol{\sigma}$, respectively. The notations $(x, y, z) = (x_1, x_2, x_3)$ for Cartesian coordinates and $(u, v, w) = (u_1, u_2, u_3)$ for the corresponding velocity components are equivalent. Unit vectors in given coordinate directions are denoted by symbols $\mathbf{e}_x$, $\mathbf{e}_r$, etc. Partial derivatives are denoted either by subscripts or by the symbol ∂, and $\partial_i = \partial/\partial x_i$. Thus, e.g.,

$$T_x = \partial_x T = \partial_1 T = \frac{\partial T}{\partial x} = \frac{\partial T}{\partial x_1}. \tag{0.1}$$

The symbol ∇_h is the 2-D gradient operator with respect to the coordinates (x, y) or (in a few cases) the spherical colatitude and longitude (θ, ϕ). The symbols $\Re[\ldots]$ and $\Im[\ldots]$ denote the real and imaginary parts, respectively, of the bracketed quantities. A 'free-slip' surface is one on which both the normal velocity and the shear traction vanish.

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Abbreviations

1-D	one-dimensional
2-D	two-dimensional
3-D	three-dimensional
BEM	boundary-element method
BL	boundary layer
BVP	boundary-value problem
CMB	core–mantle boundary
GIA	glacial isostatic adjustment
IMP	intermediate matching principle
LHS	left-hand side
LSA	linear stability analysis
MEE	method of eigenfunction expansions
MMAE	method of matched asymptotic expansions
ODE	ordinary differential equation
PDE	partial differential equation
PMM	propagator matrix method
RHS	right-hand side
RII	reactive infiltration instability
RBC	Rayleigh–Bénard convection
RTI	Rayleigh–Taylor instability
SBT	slender-body theory
TBL	thermal boundary layer