Part I

Vlasov Weak Turbulence Theory: Electrostatic Approximation

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Nonlinear Electrostatic Equations for Collisionless Plasmas

This monograph presents a perturbative nonlinear kinetic theory of plasma turbulence, known as the *weak turbulence theory*. At the outset, it should be pointed out that this book does not include the effects of ambient magnetic field. Plasmas in real situations are usually magnetized, so that applications of the method discussed in this book will be somewhat limited, but the purpose is to lay out the fundamental methodology and conceptual foundations so that more general applications for magnetized plasmas may be developed on the basis of this book. This book also limits the discussions to spatially homogeneous plasma.

Plasma kinetic theory has a long history, and many early papers can be found in the literature that discuss the perturbative nonlinear kinetic theory of plasma turbulence – see, for example, papers by Vedenov and Velikhov (1962); Kovrizhnykh and Tsytovich (1964, 1965); Kovrizhnykh (1965); Gorbunov and Silin (1965); Gorbunov et al. (1965); Tsytovich (1967); Rogister and Oberman (1968, 1969), to name just several. These are merely sample papers, among those that personally influenced the author of this book.

If one is interested in the general background on plasma kinetic theory, there are some excellent early monographs, among which may be, for instance, those by Montgomery and Tidman (1964); Kadomtsev (1965); Klimontovich (1967, 1982); Pitaevskii and Lifshitz (1981); Sagdeev and Galeev (1969); Tsytovich (1970, 1977a,b); Davidson (1972); Ichimaru (1973); Krall and Trivelpiece (1973); Akhiezer et al. (1975); Hasegawa (1975); Kaplan and Tsytovich (1973); Sitenko (1967, 1982); Melrose (1980a, 1986); Nicholson (1983); Alexandrov et al. (1984); Chen (1987), etc. This list is incomplete, but they represent some standard works that treat the foundations of plasma kinetic theory and/or weak plasma turbulence theory. More recent books are also available. See, for example, those by Musher et al. (1995); Sitenko and Malnev (1995); Treumann and Baumjohann (1997); Tsytovich (1995); Kono and Škorić (2010); Diamond et al. (2010), etc., which deal with the subject of plasma kinetic theory and nonlinear phenomena.

So, as the readers may appreciate, there is an abundance of resources on the topic of plasma kinetic theory, and one may ask why another book? The rationale for this book is as follows: Discussions of nonlinear plasma theories, particularly those concerning the weak turbulence theory found in many of the above-cited works, are sometimes not so easy to follow, especially for young researchers. Moreover, many of the monographs cover wide-ranging topics with generally brief descriptions for each subject area without going much into in-depth discussions. It is the purpose of this book to focus only on the kinetic theory of weak plasma turbulence, but to present the detailed fundamental discussions and derivations as clearly as possible, without sacrificing the intermediate mathematical steps. Talking of the latter, many authors omit too many intermediate steps, which can be a source of much frustrations for young scientists. This book does not spare the readers the mathematical details. This strategy means that some materials in the book can be a bit lengthy, and casual readers may get lost in the maths. However, if one approaches the material with enough patience, he or she will be rewarded with the intimate knowledge on how the weak turbulence theory actually works, what are the essential assumptions behind the theory, and so forth. Owing to the space devoted to mathematical details, some standard topics often included in the textbooks and monographs on nonlinear plasma theory are left out. For instance, parametric instabilities, solitary wave theory, coherent nonlinear structure formation in plasma, etc., are not covered in this book.

This book is intended for advanced undergraduate, graduate students, or young researchers who are already familiar with the introductory level of plasma kinetic theory, but wishing to familiarize themselves with a more in-depth understanding on nonlinear theory of weak plasma turbulence. In spite of this, this book expounds on foundational principles at the conceptual level as much as possible without assuming too much prior knowledge on the part of the readers.

1.1 Preamble: Fundamental Concepts

We are interested in physical phenomena that are described as turbulent, which loosely means physical quantities that are fluctuating in space and time. In order to characterize such fluctuations, we employ statistical methods and concepts. That is, we deal with averages in time, space, or over hypothetical collection of different possible states called the *ensemble*. One is particularly interested in how fluctuating quantities measured in two or more different times or in two or more different spatial locations are correlated. We begin by considering many-body correlations associated with fluctuating physical quantities, and the spectral transformation of such quantities in space and time.

The statistical correlation is an important concept that characterizes the nature of turbulence. Suppose that one measures a particular physical quantity, say velocity or electromagnetic field, in a turbulent medium at a given time. Suppose also that one measures the same quantity at another time separated by an interval. If one repeats such series of measurements over and over again, then if the physical quantities are uncorrelated, that is, if there is no cause and effect relationship between the two measurements, then on average, the product of two measurements made at two different time intervals may be zero, since by the very nature of turbulence, velocity or field may have random directions. On the other hand, if the first measurement affects the second measurement because there exists an underlying cause-and-effect relationship, then the average of the products may be finite. A systematic way to characterize how the statistical average of the products of physical quantities, or equivalently, their correlation function, behaves in space and time can thus be useful for understanding and characterizing the turbulence. Consequently, in this book we will be concerned with the description of how the statistical average of the correlation of fluctuating (i.e., turbulent) quantity, $\langle \delta a^2 \rangle$, dynamically evolves. Here, δa represents any dynamical quantity, and the symbol $\langle \cdots \rangle$ denotes the statistical average.

The convention adopted in this book for the definition of spatial Fourier transformation and its inverse is

$$f_{\mathbf{k}} = (2\pi)^{-3} \int d\mathbf{r}\ f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}, \qquad f(\mathbf{r}) = \int d\mathbf{k}\ f_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{1.1}$$

Here $f(\mathbf{r})$ is any physical quantity, which is a function of spatial coordinate $\mathbf{r}$, and which is bounded in space. The Fourier transformation of a product of two functions is represented by the convolution

$$(2\pi)^{-3} \int d\mathbf{r}\ f(\mathbf{r})\, g(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}} = \int d\mathbf{k}'\ f_{\mathbf{k}'}\, g_{\mathbf{k}-\mathbf{k}'} = \int d\mathbf{k}'\ f_{\mathbf{k}-\mathbf{k}'}\, g_{\mathbf{k}'}. \tag{1.2}$$

The proof of this "convolution theorem" is straightforward. All one has to do is to insert for $f(\mathbf{r})$ and $g(\mathbf{r})$, their respective Fourier transformations, and make use of the well-known delta function identity

$$\int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} = \delta(\mathbf{k}). \tag{1.3}$$

Fourier transformation of a function $f(\mathbf{r},t)$ in both space and time can be defined by

$$\begin{aligned} f_{\mathbf{k},\omega} &= (2\pi)^{-4} \int d\mathbf{r} \int dt\ f(\mathbf{r},t)\, e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}, \\ f(\mathbf{r},t) &= \int d\mathbf{k} \int d\omega\ f_{\mathbf{k},\omega}\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}. \end{aligned} \tag{1.4}$$

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Convolution theorem for the spatio-temporal Fourier transformation is

$$\begin{aligned}(2\pi)^{-4}\int d\mathbf{r}\int dt\; f(\mathbf{r},t)\,g(\mathbf{r},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}\\ =\int d\mathbf{k}'\int d\omega'\; f_{\mathbf{k}',\omega'}\,g_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\\ =\int d\mathbf{k}'\int d\omega'\; f_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\,g_{\mathbf{k}',\omega'}.\end{aligned} \tag{1.5}$$

When the angular frequency ω satisfies the dispersion relation $\omega = \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}}$, that is, when (generally complex) ω is a function of $\mathbf{k}$, then the Fourier representation of function $f(\mathbf{r},t)$ can be re-expressed by virtue of the fact that we may write the spectral amplitude as

$$f_{\mathbf{k},\omega} = f_{\mathbf{k}}\delta(\omega - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}), \tag{1.6}$$

or

$$f(\mathbf{r},t) = \int d\mathbf{k}\; f_{\mathbf{k}}\;\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}}t + \gamma_{\mathbf{k}}t). \tag{1.7}$$

If $f(\mathbf{r},t)$ is real then obviously $f^*(\mathbf{r},t) = f(\mathbf{r},t)$, where the asterisk $*$ represents the complex conjugate. From this it follows that

$$\int d\mathbf{k}\; f_{\mathbf{k}}^*\;\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega_{\mathbf{k}}t) = \int d\mathbf{k}\; f_{\mathbf{k}}\;\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}}t), \tag{1.8}$$

which leads to the following symmetry relations:

$$f_{\mathbf{k}}^* = f_{-\mathbf{k}}, \qquad \omega_{-\mathbf{k}} = -\omega_{\mathbf{k}}, \qquad \gamma_{-\mathbf{k}} = \gamma_{\mathbf{k}}. \tag{1.9}$$

Let $\delta f(\mathbf{r},t)$ represent a fluctuating quantity whose ensemble average is zero:

$$\langle \delta f(\mathbf{r},t)\rangle = 0. \tag{1.10}$$

In our notation, any quantity preceded by δ indicates that this quantity is fluctuating in space and time, that is, turbulent. By "ensemble average" we may mean an average over phase, space, or time. Or it could mean an average over all possible configurations. Turbulence is called "homogeneous" if the spatial dependence of the two-body correlation is only upon the relative distance,

$$\langle \delta f(\mathbf{r},t)\,\delta f(\mathbf{r}',t)\rangle = \langle \delta f^2\rangle_{\mathbf{r}-\mathbf{r}',t,t'} = \langle \delta f^2\rangle_{\mathbf{r}'-\mathbf{r},t,t'}, \tag{1.11}$$

and "stationary" if the temporal two-body correlation is a function of relative time difference,

$$\langle \delta f(\mathbf{r},t)\,\delta f(\mathbf{r},t')\rangle = \langle \delta f^2\rangle_{\mathbf{r},\mathbf{r}',t-t'} = \langle \delta f^2\rangle_{\mathbf{r},\mathbf{r}',t'-t}. \tag{1.12}$$

Thus, for homogeneous and stationary turbulence the two-body correlation function is given by

$$\langle \delta f(\mathbf{r},t)\,\delta f(\mathbf{r}',t')\rangle = \langle \delta f^2\rangle_{\mathbf{r}-\mathbf{r}',t-t'}. \tag{1.13}$$

It should be noted that not all fluctuating quantities in nature satisfy the zero ensemble average property (1.10). Physical processes whose fluctuations satisfy (1.10) are called "incoherent" phenomena, while "coherent" processes may be associated with a nonvanishing ensemble average. For incoherent processes different phases are uncorrelated such that when averaged over them, the result vanishes; hence, such processes are characterized by the zero ensemble average property specified by (1.10).

In a similar way, the three-body correlation function for homogeneous and stationary turbulence is a function of distances between any two points, say $(\mathbf{r},t)$ and $(\mathbf{r}',t')$, among three points $(\mathbf{r},t)$, $(\mathbf{r}',t')$, $(\mathbf{r}'',t'')$, in coordinate-time space:

$$\langle \delta f(\mathbf{r},t)\,\delta f(\mathbf{r}',t')\,\delta f(\mathbf{r}'',t'')\rangle = \langle \delta f^3\rangle_{\mathbf{r}-\mathbf{r}',\mathbf{r}-\mathbf{r}'';t-t',t-t''}. \tag{1.14}$$

The four-body correlation function for homogeneous and stationary turbulence can be defined likewise:

$$\begin{aligned}
&\langle \delta f(\mathbf{r},t)\,\delta f(\mathbf{r}',t')\,\delta f(\mathbf{r}'',t'')\,\delta f(\mathbf{r}''',t''')\rangle \\
&\quad = \langle \delta f^2\rangle_{\mathbf{r}-\mathbf{r}';t-t'}\,\langle \delta f^2\rangle_{\mathbf{r}''-\mathbf{r}''';t''-t'''} + \langle \delta f^2\rangle_{\mathbf{r}-\mathbf{r}'';t-t''}\,\langle \delta f^2\rangle_{\mathbf{r}'-\mathbf{r}''';t'-t'''} \\
&\qquad + \langle \delta f^2\rangle_{\mathbf{r}-\mathbf{r}''';t-t'''}\,\langle \delta f^2\rangle_{\mathbf{r}'-\mathbf{r}'';t'-t''} \\
&\qquad + \langle \delta f^4\rangle_{\mathbf{r}-\mathbf{r}',\mathbf{r}'-\mathbf{r}'',\mathbf{r}''-\mathbf{r}''';t-t',t'-t''',t''-t'''}.
\end{aligned} \tag{1.15}$$

Let us represent the two-body correlation function in spectral form:

$$\begin{aligned}
\langle \delta f(\mathbf{r},t)\,\delta f(\mathbf{r}',t')\rangle &= \int d\mathbf{k}\int d\omega\;\langle \delta f^2\rangle_{\mathbf{k},\omega}\;e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')} \\
&= \int d\mathbf{k}\int d\omega\int d\mathbf{k}'\int d\omega'\;\langle \delta f_{\mathbf{k},\omega}\,\delta f_{\mathbf{k}',\omega'}\rangle\;e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'-i\omega t-i\omega' t'},
\end{aligned} \tag{1.16}$$

where in the second line we have made use of the spectral representations for individual functions $\delta f(\mathbf{r},t)$ and $\delta f(\mathbf{r}',t')$. From this, it is seen that the equality can be obtained if the following condition is satisfied:

$$\langle \delta f_{\mathbf{k},\omega}\,\delta f_{\mathbf{k}',\omega'}\rangle = \delta(\mathbf{k}+\mathbf{k}')\;\delta(\omega+\omega')\;\langle \delta f^2\rangle_{\mathbf{k},\omega}. \tag{1.17}$$

If we write the spectral component $\delta f_{\mathbf{k},\omega}$ with an explicit phase factor,

$$\delta f_{\mathbf{k},\omega} = \hat{f}_{\mathbf{k},\omega} e^{i\phi_{\mathbf{k},\omega}}, \tag{1.18}$$

where $\phi_{\mathbf{k},\omega}$ represents the phase, then we have

$$\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'} \rangle = \langle \hat{f}_{\mathbf{k},\omega}\, \hat{f}_{\mathbf{k}',\omega'} e^{i\phi_{\mathbf{k},\omega}+i\phi_{\mathbf{k}',\omega'}} \rangle. \tag{1.19}$$

For homogeneous and stationary turbulence the phase is assumed to be random (or uncorrelated). As such, the ensemble average over random phases becomes nonzero only if

$$\phi_{\mathbf{k},\omega} + \phi_{\mathbf{k}',\omega'} = 0, \tag{1.20}$$

which can be satisfied under the assumption that, if for $\mathbf{k} = -\mathbf{k}'$ and $\omega = -\omega'$, the following is also satisfied:

$$\phi_{-\mathbf{k},-\omega} = -\phi_{\mathbf{k},\omega}. \tag{1.21}$$

This is but the rephrasing of condition (1.17). The assumption of homogeneous and stationary turbulence is thus equivalent to the "random phase approximation." In short, the property

$$\langle \delta f^2 \rangle_{\mathbf{k},\omega} = \langle \delta f_{\mathbf{k},\omega}\, \delta f_{-\mathbf{k},-\omega} \rangle \tag{1.22}$$

is a useful spectral characteristic for homogeneous and stationary turbulence, or equivalently, fluctuations with random phases.

Next, consider the three-body correlation, which we may write as

$$\begin{aligned}
\langle \delta f(\mathbf{r},t)\, \delta f(\mathbf{r}',t')\, \delta f(\mathbf{r}'',t'') \rangle &= \int d\mathbf{k} \int d\omega \int d\mathbf{k}' \int d\omega'\; \langle \delta f^3 \rangle_{\mathbf{k},\omega;\mathbf{k}',\omega'} \\
&\quad \times e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')+i\mathbf{k}'\cdot(\mathbf{r}'-\mathbf{r}'')-i\omega(t-t')-i\omega'(t'-t'')} \\
&= \int d\mathbf{k} \int d\omega \int d\mathbf{k}' \int d\omega' \int d\mathbf{k}'' \int d\omega'' \\
&\quad \times \langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''} \rangle \\
&\quad \times e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'+i\mathbf{k}''\cdot\mathbf{r}''-i\omega t-i\omega' t'-i\omega'' t''}.
\end{aligned} \tag{1.23}$$

From this we obtain the identity

$$\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''} \rangle = \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\; \delta(\omega+\omega'+\omega'')\; \langle \delta f^3 \rangle_{\mathbf{k},\omega;\mathbf{k}+\mathbf{k}',\omega+\omega'}. \tag{1.24}$$

A similar analysis can be carried out for the four-body correlation. The derivation is tedious but straightforward, and is thus omitted.

We summarize the general properties of the many-body correlations, or many-body cumulants for homogeneous and stationary turbulence:

$$\begin{aligned}
\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'} \rangle &= \delta(\mathbf{k}+\mathbf{k}')\; \delta(\omega+\omega') \langle \delta f^2 \rangle_{\mathbf{k},\omega}, \\
\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''} \rangle &= \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\delta(\omega+\omega'+\omega'') \langle \delta f^3 \rangle_{\mathbf{k},\omega;\mathbf{k}+\mathbf{k}',\omega+\omega'},
\end{aligned}$$

$$\begin{aligned}\langle \delta f_{\mathbf{k},\omega}\, \delta f_{\mathbf{k}',\omega'}\, \delta f_{\mathbf{k}'',\omega''}\, \delta f_{\mathbf{k}'''\omega'''} \rangle = {} & \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''')\, \delta(\omega+\omega'+\omega''+\omega''') \\ & \times [\delta(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega')\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}\, \langle \delta f^2 \rangle_{\mathbf{k}'',\omega''} \\ & + \delta(\mathbf{k}+\mathbf{k}'')\, \delta(\omega+\omega'')\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}\, \langle \delta f^2 \rangle_{\mathbf{k}',\omega'} \\ & + \delta(\mathbf{k}'+\mathbf{k}'')\, \delta(\omega'+\omega'')\, \langle \delta f^2 \rangle_{\mathbf{k},\omega}\, \langle \delta f^2 \rangle_{\mathbf{k}',\omega'} \\ & + \langle \delta f^4 \rangle_{\mathbf{k},\omega;\, \mathbf{k}+\mathbf{k}',\omega+\omega';\, \mathbf{k}+\mathbf{k}'+\mathbf{k}'',\omega+\omega'+\omega''}]. \end{aligned} \tag{1.25}$$

An important consequence of this result is that an ensemble average of two fluctuating quantities δf and δg, where they are related to each other, can be expressed in terms of their spectral counterparts as follows:

$$\langle \delta f(\mathbf{r},t)\, \delta g(\mathbf{r},t) \rangle = \int d\mathbf{k} \int d\omega\, \langle \delta f_{\mathbf{k},\omega}\, \delta g_{-\mathbf{k},-\omega} \rangle. \tag{1.26}$$

1.2 Electrostatic Vlasov Equation

A simple and intuitive definition of *plasma* is that it is an *ionized gas*. Individual electrons and ions that make up the plasma interact through collective electromagnetic force. Collective behavior of a plasma is described by a statistical means. In this book we are concerned with a fully ionized plasma. For partially ionized plasma, atomic processes such as the recombination and collisions between charged particles and neutrals cannot be ignored, which complicate the matter. Vlasov equation (Vlasov, 1938) describes the statistical property of a plasma governed by collective processes. The system under consideration is a spatially uniform plasma made of single-species ions (protons) and electrons, and there is no net electric or magnetic field. We also assume zero average charge or current in the system. If we make the simplifying approximation that the plasma particles interact primarily through electrostatic field, then the dynamics can be described by the Vlasov–Poisson system of equations

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}} \right) f_a = 0, \\ \nabla\cdot\mathbf{E} = 4\pi \sum_a e_a \int d\mathbf{v}\, f_a, \end{aligned} \tag{1.27}$$

where e_a and m_a are charge and mass of species a $(= e, i)$ for electrons and ions ($e_a = e$ for protons and $e_a = -e$ for electrons). The one-particle distribution function $f_a(\mathbf{r},\mathbf{v},t)$ is the probability density of finding a collection of plasma particles of species a, at a particular state in phase space $(\mathbf{r},\mathbf{v})$ at a given time t. Consequently, if we integrate $f_a(\mathbf{r},\mathbf{v},t)$ over $\mathbf{v}$, or equivalently, if we collect all possible configuration in velocity space, then the result becomes the density of charged particle species labeled a,

$$\rho_a(\mathbf{r},t) = \int d\mathbf{v}\, f_a(\mathbf{r},\mathbf{v},t). \tag{1.28}$$

Multiplying the charge e_a and summing over all charged particle species leads to the total charge density

$$\rho(\mathbf{r},t) = \sum_a e_a \rho_a(\mathbf{r},t). \tag{1.29}$$

Since $f_a(\mathbf{r},\mathbf{v},t)$ is the probability density, it is normalized to the ambient charged particle number density n_a,

$$\frac{1}{\mathscr{V}} \int d\mathbf{r} \int d\mathbf{v}\; f_a(\mathbf{r},\mathbf{v},t) = n_a, \tag{1.30}$$

where $\mathscr{V}$ is the volume of the system. That is, if we collect all possible configurations in velocity space at a given time, and integrate over the entire volume under consideration and divide by $\mathscr{V}$, that is, take the spatial average, then the result should be the total number of particles per volume, $n_a = N_a/\mathscr{V}$, or equivalently, the ambient density. Since in the absence of source or sink, plasma particles cannot be created or annihilated (that is, no recombination into neutrals or reionization), the one-particle distribution function must be conserved. Hence,

$$\frac{df_a}{dt} = \left(\frac{\partial}{\partial t} + \dot{\mathbf{r}}\cdot\nabla + \dot{\mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f_a = 0. \tag{1.31}$$

By virtue of the equation of motion,

$$\dot{\mathbf{r}} = \mathbf{v} \text{ and } \dot{\mathbf{v}} = \frac{e_a}{m_a}\mathbf{E}, \tag{1.32}$$

we obtain the Vlasov equation in (1.27). Because of the charge neutrality condition, the ambient density is the same for both ions and electrons,

$$n_e = n_i = n. \tag{1.33}$$

Let us separate the physical quantities into average and fluctuating parts. The average particle distribution function is independent of the spatial coordinate $\mathbf{r}$ since we assume uniform plasma, and there is no average electric field, so that we may write

$$\begin{aligned} f_a(\mathbf{r},\mathbf{v},t) &= n_a F_a(\mathbf{v},t) + \delta f_a(\mathbf{r},\mathbf{v},t), \\ \mathbf{E}(\mathbf{r},t) &= \delta\mathbf{E}(\mathbf{r},t), \end{aligned} \tag{1.34}$$

where δ represents fluctuating quantities whose phases are supposed to be random. When averaged over their phases, these quantities vanish. In (1.34) we have introduced a normalized one-particle distribution function $F_a(\mathbf{v},t)$ $[\int d\mathbf{v}\, F_a(\mathbf{v},t) = 1]$. Inserting (1.34) back into the coupled Vlasov–Poisson equation, we obtain