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Excerpt
More Information

1 Fundamental Principles

1.1 Classification of Problems in Structural Dynamics

As indicated in the list that follows, the study of structural dynamics – and books about the subject – can be organized and classified according to various criteria. This book follows largely the first of these classifications, and with the exceptions of nonlinear systems, addresses all of these topics.

(a) By the number of degrees of freedom:

- **Single DOF**
- **Multiple DOFs**
  - lumped mass (discrete) system (finite DOF)
  - continuous systems (infinitely many DOF)

Discrete systems are characterized by systems of ordinary differential equations, while continuous systems are described by systems of partial differential equations.

(b) By the linearity of the governing equations:

- **Linear systems** (linear elasticity, small motions assumption)
- **Nonlinear systems**
  - conservative (elastic) systems
  - nonconservative (inelastic) systems

(c) By the type of excitation:

- **Free vibrations**
- **Forced vibrations**
  - structural loads / seismic loads
    - periodic
      - harmonic
      - nonharmonic
    - transient
      - deterministic excitation
      - random excitation
        - stationary
        - nonstationary

(d) By the type of mathematical problem:

$$\begin{cases}\textbf{Static} \rightarrow \text{boundary value problems} \\ \textbf{Dynamic} \quad \begin{cases}\text{eigenvalue problems (free vibrations)} \\ \text{initial value problem, propagation problem (waves)}\end{cases}\end{cases}$$

(e) By the presence of energy dissipating mechanisms:

$$\begin{cases}\textbf{Undamped vibrations} \\ \textbf{Damped vibrations} \quad \begin{cases}\text{viscous damping} \\ \text{hysteretic damping} \\ \text{Coulomb damping} \\ \text{etc.}\end{cases}\end{cases}$$

1.2 Stress–Strain Relationships

1.2.1 Three-Dimensional State of Stress–Strain

$$\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \varepsilon_z\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1\end{bmatrix}\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \sigma_z\end{Bmatrix}, \qquad \begin{aligned} &E = 2G(1+\nu) = \text{ Young's modulus} \\ &\nu = \text{Poisson's ratio}\end{aligned} \tag{1.1}$$

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \sigma_z\end{Bmatrix} = \frac{2G}{1-2\nu}\begin{bmatrix}1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu\end{bmatrix}\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \varepsilon_z\end{Bmatrix}, \qquad \lambda = \frac{2G\,\nu}{1-2\nu} = \text{ Lamé constant} \tag{1.2}$$

$$\tau_{xy} = G\gamma_{xy}, \qquad \tau_{xz} = G\gamma_{xz}, \qquad \tau_{yz} = G\gamma_{yz} \tag{1.3}$$

1.2.2 Plane Strain

$$\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y\end{Bmatrix} = \frac{1}{2G}\begin{bmatrix}1-\nu & -\nu \\ -\nu & 1-\nu\end{bmatrix}\begin{Bmatrix}\sigma_x \\ \sigma_y\end{Bmatrix} \tag{1.4}$$

$$\begin{Bmatrix}\sigma_x \\ \sigma_y\end{Bmatrix} = \frac{2G}{1-2\nu}\begin{bmatrix}1-\nu & \nu \\ \nu & 1-\nu\end{bmatrix}\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y\end{Bmatrix} \tag{1.5}$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \qquad \sigma_z = \nu(\sigma_x + \sigma_y) = \lambda(\varepsilon_x + \varepsilon_y), \qquad \tau_{xy} = G\gamma_{xy} \tag{1.6}$$

1.2.3 Plane Stress

$$\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu \\ -\nu & 1\end{bmatrix}\begin{Bmatrix}\sigma_x \\ \sigma_y\end{Bmatrix} = \frac{1}{2G(1+\nu)}\begin{bmatrix}1 & -\nu \\ -\nu & 1\end{bmatrix}\begin{Bmatrix}\sigma_x \\ \sigma_y\end{Bmatrix} \tag{1.7}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = \frac{2G}{1-\nu} \begin{Bmatrix} 1 & \nu \\ \nu & 1 \end{Bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \end{Bmatrix} \tag{1.8}$$

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0, \qquad \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right), \qquad \tau_{xy} = G\gamma_{xy} \tag{1.9}$$

1.2.4 Plane Stress versus Plane Strain: Equivalent Poisson's Ratio

We explore here the possibility of defining a *plane strain* system with Poisson's ratio $\tilde{\nu}$ such that it is equivalent to a *plane stress* system with Poisson's ratio ν while having the same shear modulus G. Comparing the stress–strain equations for plane stress and plane strain, we see that this would require the simultaneous satisfaction of the two equations

$$\frac{1-\tilde{\nu}}{1-2\tilde{\nu}} = \frac{1}{1-\nu} \qquad \text{and} \qquad \frac{\tilde{\nu}}{1-2\tilde{\nu}} = \frac{\nu}{1-\nu} \tag{1.10}$$

which are indeed satisfied if

$$\boxed{\tilde{\nu} = \frac{\nu}{1+\nu}} \quad \text{plane strain ratio } \tilde{\nu} \text{ that is equivalent to the plane-stress ratio } \nu \tag{1.11}$$

Hence, it is always possible to map a plane-stress problem into a plane-strain one.

1.3 Stiffnesses of Some Typical Linear Systems

Notation

E = Young's modulus
G = shear modulus
ν = Poisson's ratio
A = cross section
A_s = shear area
I = area moment of inertia about bending axis
L = length

Linear Spring

Longitudinal spring k_x

Helical (torsional) spring $k = \dfrac{G\,d^4}{8\,n\,D^3}$

where

d = wire diameter
D = mean coil diameter
n = number of turns

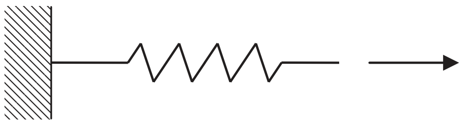

Figure 1.1

Member stiffness matrix = $\mathbf{K} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix}$

Rotational Spring

Rotational stiffness k_θ

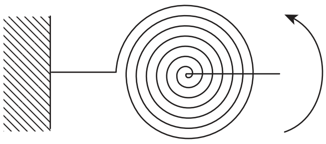

Figure 1.2

Floating Body

Buoyancy stiffness

$k_z = \rho_w \, g \, A_w$ Heaving (up and down)

$k_{\theta x} = \rho_w \, g \, I_w$ Rolling about x-axis (small rotations)

$k_{x,y} = 0$ Lateral motion

in which ρ_w = mass density of water; g = acceleration of gravity; A_w = horizontal cross section of the floating body at the level of the water line; and I_w= area moment of inertia of A_w with respect to the rolling axis (x here). If the floating body's lateral walls in contact with the water are not vertical, then the heaving stiffness is valid only for small vertical displacements (i.e., displacements that cause only small changes in A_w). In addition, the rolling stiffness is just an approximation for small rotations, even with vertical walls. Thus, the buoyancy stiffnesses given earlier should be interpreted as tangent stiffnesses or incremental stiffnesses.

Observe that the weight of a floating body equals that of the water displaced.

Cantilever Shear Beam

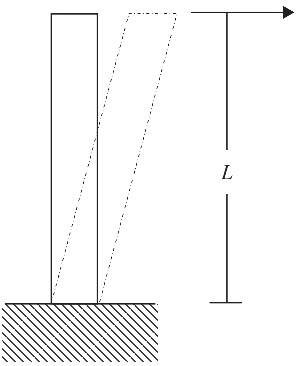

Figure 1.3

A shear beam is infinitely stiff in rotation, which means that no rotational deformation exists. However, a free (i.e., unrestrained) shear beam may rotate as a rigid body. After deformation, sections remain parallel. If the cantilever beam is subjected to a moment at its free end, the beam will remain undeformed. The moment is resisted by an equal and opposite moment at the base. If a force P acts at an elevation $a \le L$ above the base, the lateral displacement increases linearly from zero to $u = Pa / GA_s$ and remains constant above that elevation.

Typical shear areas : Rectangular section $A_s = \frac{5}{6} A$

Ring $A_s = \frac{1}{2} A$

Stiffness as perceived at the top:

Axial stiffness $k_z = \dfrac{EA}{L}$

Transverse stiffness $k_x = \dfrac{GA_s}{L}$

Rotational stiffness $k_\varphi = \infty$

Cantilever Bending Beam

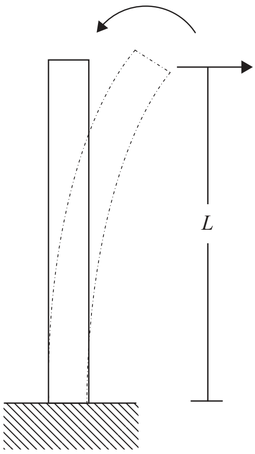

Figure 1.4

Stiffnesses as perceived at the top:

Axial stiffness $k_z = \dfrac{EA}{L}$

Transverse stiffness (rotation positive counterclockwise):

(a) Free end $k_x = \dfrac{3EI}{L^3}$ (rotation unrestrained)

$k_\phi = \dfrac{EI}{L}$ (translation unrestrained)

(b) Constrained end $k_{xx} = \dfrac{12EI}{L^3}$ $k_{x\varphi} = \dfrac{6EI}{L^2}$

$$k_{\varphi x} = \frac{6EI}{L^2} \qquad k_{\varphi\varphi} = \frac{4EI}{L}$$

$$\mathbf{K}_{\mathrm{BB}} = \frac{EI}{L^3}\begin{Bmatrix} 12 & 6L \\ 6L & 4L^2 \end{Bmatrix}$$

Notice that carrying out the static condensations $k_x = k_{xx} - k_{x\varphi}^2 / k_{\varphi\varphi}$ and $k_\varphi = k_{\varphi\varphi} - k_{x\varphi}^2 / k_{xx}$ we recover the stiffness k_x, k_φ for the two cases when the loaded end is free to rotate or translate.

Transverse Flexibility of Cantilever Bending Beam

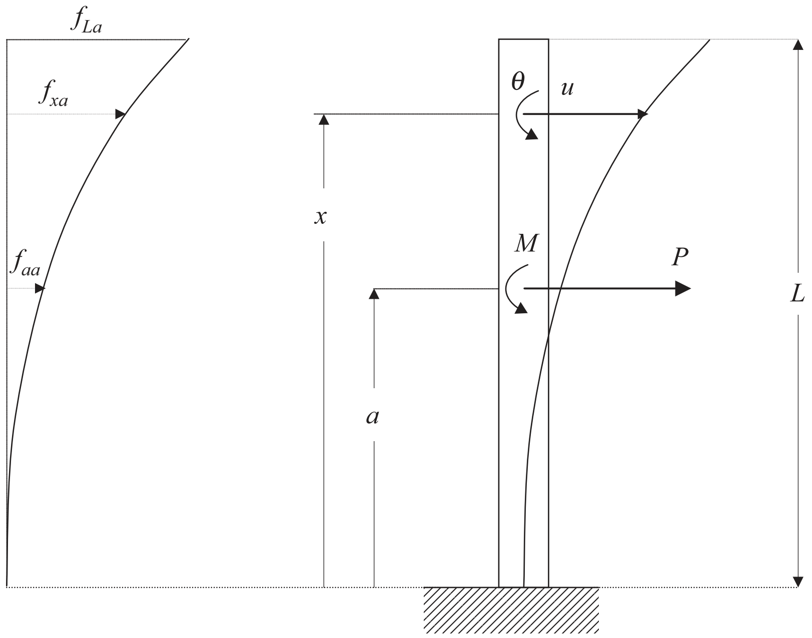

Figure 1.5

A lateral load P and a *counterclockwise* moment M are applied simultaneously to a cantilever beam at some arbitrary distance a from the support (Figure 1.5). These cause in turn a transverse displacement u and a counterclockwise rotation θ at some other distance x from the support. These are

$$u(x) = \frac{P}{6EI}x^2(3a - x) - \frac{M}{2EI}x^2 \qquad x \le a$$

$$\theta(x) = -\frac{P}{2EI}x(2a - x) + \frac{M}{EI}x \qquad x \le a$$

and

$$u(x) = \frac{P}{6EI} a^2 (3x - a) - \frac{M}{2EI} a (2x - a) \qquad x \ge a$$

$$\theta(\xi) = -\frac{P}{2EI} a^2 + \frac{M}{EI} a \qquad x \ge a$$

Transverse Flexibility of Simply Supported Bending Beam

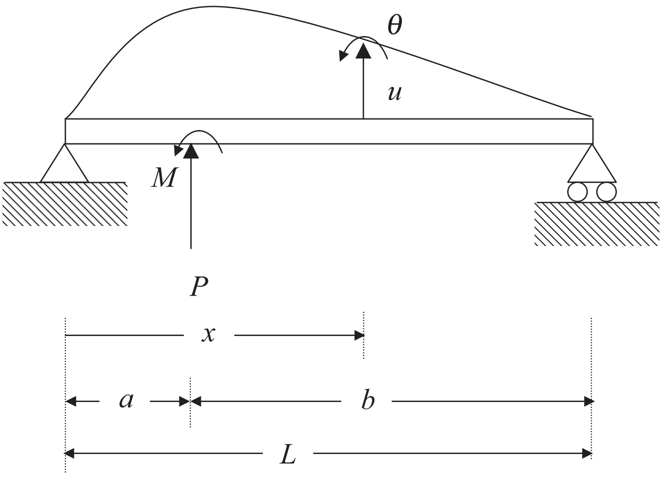

Figure 1.6

A transverse load P and a counterclockwise moment M are applied to a simply supported beam at some arbitrary distance a from the support (Figure 1.6). These cause in turn a transverse displacement u and rotation θ (positive up and counterclockwise, respectively) at some other distance x from the support. Defining the dimensionless coordinates $\alpha = a/L$, $\beta = 1 - \alpha$ and $\xi = x/L$, the observed displacement and rotation are:

$$u(x) = \frac{P L^3}{6 EI} \beta\xi \left(1 - \beta^2 - \xi^2\right) + \frac{M L^2}{6 EI} \xi \left(\xi^2 + 3\beta^2 - 1\right), \qquad \xi \le \alpha$$

$$\theta(x) = \frac{P L^2}{6 EI} \beta \left(1 - \beta^2 - 3\xi^2\right) + \frac{M L}{6 EI} \left(3\xi^2 + 3\beta^2 - 1\right), \qquad \xi \le \alpha$$

and

$$u(x) = \frac{P L^3}{6 EI} \alpha (1 - \xi) \left[1 - \alpha^2 - (1 - \xi)^2\right] + \frac{M L^2}{6 EI} (1 - \xi) \left[1 - 3\alpha^2 - (1 - \xi)^2\right] \qquad \xi \ge \alpha$$

$$\theta(x)=\frac{PL^2}{6EI}\alpha\left[3(1-\xi)^2+\alpha^2-1\right]+\frac{ML}{6EI}\left[3(1-\xi)^2+3\alpha^2-1\right] \qquad \xi\ge\alpha$$

In particular, at $\xi=\alpha$

$$u(\alpha)=\frac{PL^3}{3EI}\alpha^2\beta^2+\frac{ML^2}{3EI}\alpha\beta(\beta-\alpha)$$

$$\theta(\alpha)=\frac{PL^2}{3EI}\alpha\beta(\beta-\alpha)+\frac{ML}{3EI}(1-3\alpha\beta)$$

Transverse stiffness at $x=a$ $\qquad k=\dfrac{3EIL}{a^2b^2}$ $\qquad$ (rotation permitted)

Special case : $a=b=L/2$ $\qquad k=\dfrac{48EI}{L^3}$ $\qquad$ (rotation permitted)

Stiffness and Inertia of Free Beam with Shear Deformation Included

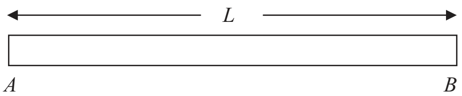

Figure 1.7

Consider a free, homogeneous bending beam AB of length L that lies horizontally in the vertical plane $x-y$ and deforms in that plane, as shown in Figure 1.7. It has mass density ρ, Poison's ratio ν, shear modulus G, Young's modulus $E=2G(1+\nu)$, area-moment of inertia I_z about the horizontal bending axis z (i.e., bending in the plane), cross section A_x, and shear area A_{sy} for shearing in the transverse y direction. In the absence of further information about shear deformation, one can choose either $A_{sy}=A_x$ or $A_{sy}=\infty$. Define

$$m=\rho A_x L,\quad j_z=\rho I_z/L=m\left(\frac{R_z}{L}\right)^2,\quad R_z=\sqrt{\frac{I_z}{A_x}},\quad \phi_z=\frac{12EI_z}{GA_{sy}L^2}=24(1+\nu)\frac{I_z}{A_{sy}L^2}$$

then from the theory of finite elements we obtain the *bending* stiffness matrix $\mathbf{K}_B$ for a bending beam with shear deformation included, together with the consistent bending mass matrix $\mathbf{M}_B$, which accounts for both translational as well as rotational inertia (rotations positive counterclockwise):

$$\mathbf{K}_B=\frac{EI_z}{(1+\phi_z)L^3}\begin{Bmatrix}12 & 6L & -12 & 6L\\ 6L & (4+\phi_z)L^2 & -6L & (2-\phi_z)L^2\\ -12 & -6L & 12 & -6L\\ 6L & (2-\phi_z)L^2 & -6L & (4+\phi_z)L^2\end{Bmatrix}$$

$$\mathbf{M}_B = \frac{m}{420}\begin{Bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{Bmatrix} + \frac{j_z}{30}\begin{Bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{Bmatrix}$$

On the other hand, the axial degrees of freedom (when the member acts as a column) have *axial* stiffness and mass matrices

$$\mathbf{K}_A = \frac{EA_x}{L}\begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}, \qquad \mathbf{M}_A = \frac{\rho ab}{6}\begin{Bmatrix} 2 & 1 \\ 1 & 2 \end{Bmatrix}$$

The *local* stiffness and mass matrices $\mathbf{K}_L, \mathbf{M}_L$ of a beam column are constructed by appropriate combinations of the bending and axial stiffness and mass matrices. These must be rotated appropriately when the members have an arbitrary orientation, after which we obtain the *global* stiffness and mass matrices.

Inhomogeneous, Cantilever Bending Beam

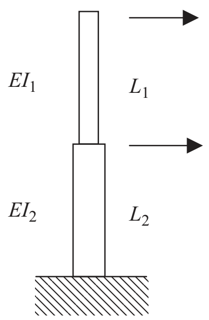

Figure 1.8

Active DOF are the two transverse displacements at the top ($j = 1$) and the junction ($j = 2$). Rotations are passive (slave) DOF.

Define the member stiffnesses $S_j = \dfrac{3EI_j}{L_j^3}, \quad j = 1,2$

The elements $k_{11}, k_{12}, k_{21}, k_{22}$ of the lateral stiffness matrix are then

$$k_{11} = \frac{S_1}{1 + \frac{3S_1}{4S_2}\left(\frac{L_1}{L_2}\right)^2} \qquad k_{12} = -k_{11}\left(1 + \frac{3}{2}\frac{L_1}{L_2}\right)$$

$$k_{21} = -k_{11}\left(1 + \frac{3}{2}\frac{L_1}{L_2}\right) \qquad k_{22} = k_{11}\left[1 + \frac{S_2}{S_1} + 3\frac{L_1}{L_2} + 3\left(\frac{L_1}{L_2}\right)^2\right]$$

Simple Frame (Portal)

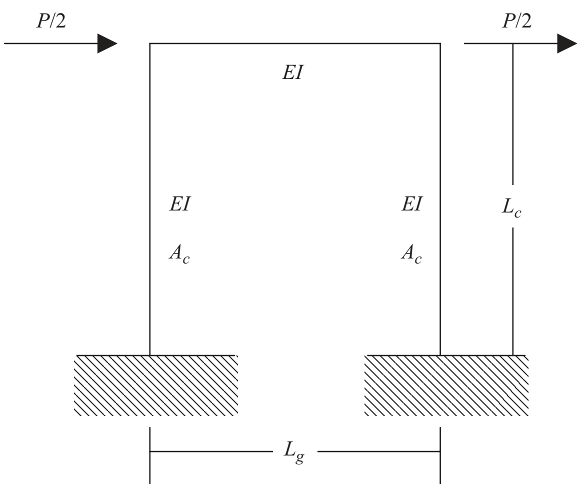

Figure 1.9

Active DOF is lateral displacement of girder.

The lateral stiffness of the frame is

$$k = \frac{24\,EI_c}{L_c^3}\left\{\frac{1+\frac{1}{6}\frac{I_c}{I_g}\frac{L_g}{L_c}+4\frac{I_c}{A_c L_g^2}}{1+\frac{2}{3}\frac{I_c}{I_g}\frac{L_g}{L_c}+16\frac{I_c}{A_c L_g^2}}\right\}$$

Includes axial deformation of columns. The girder is axially rigid.

Rigid Circular Plate on Elastic Foundation

To a first approximation and for sufficiently high excitation frequencies, a rigid circular plate on an elastic foundation behaves like a set of springs and dashpots. Here, $C_s = \sqrt{G/\rho}$ = shear wave velocity; R = radius of foundation; G = shear modulus of soil; ν = Poisson's ratio of soil; and Rocking = rotation about a horizontal axis.

Table 1.1 Stiffness and damping for circular foundation

Direction	Stiffness	Dashpot
Horizontal	$k_x = \frac{8GR}{2-\nu}$	$c_x = 0.6\frac{k_x R}{C_s}$
Vertical	$k_z = \frac{4GR}{1-\nu}$	$c_z = 0.8\frac{k_z R}{C_s}$
Rocking	$k_r = \frac{8GR^3}{3(1-\nu)}$	$c_r = 0.3\frac{k_r R}{C_s}$
Torsion	$k_t = \frac{16GR^3}{3}$	$c_t = 0.3\frac{k_t R}{C_s}$