Gödel's program for new axioms: Why, where, how and what?

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Summary. From 1931 until late in his life (at least 1970) Gödel called for the pursuit of new axioms for mathematics to settle both undecided number-theoretical propositions (of the form obtained in his incompleteness results) and undecided set-theoretical propositions (in particular CH). As to the nature of these, Gödel made a variety of suggestions, but most frequently he emphasized the route of introducing ever higher axioms of infinity. In particular, he speculated (in his 1946 Princeton remarks) that there might be a uniform (though non-decidable) rationale for the choice of the latter. Despite the intense exploration of the "higher infinite" in the last 30-odd years, no single rationale of that character has emerged. Moreover, CH still remains undecided by such axioms, though they have been demonstrated to have many other interesting set-theoretical consequences.

In this paper, I present a new very general notion of the "unfolding" closure of schematically axiomatized formal systems S which provides a uniform systematic means of expanding in an essential way both the language and axioms (and hence theorems) of such systems S. Reporting joint work with T. Strahm, a characterization is given in more familiar terms in the case that S is a basic system of non-finitist arithmetic. When reflective closure is applied to suitable systems of set theory, one is able to derive large cardinal axioms as theorems. It is an open question how these may be characterized in terms of current notions in that subject.

1. Why new axioms?

Gödel's published statements over the years (from 1931 to 1972) pointing to the need for new axioms to settle both undecided number-theoretic and set-theoretic propositions are rather well known. They are most easily cited by reference to the first two volumes of the edition of his *Collected Works*.[1] A number of less familiar statements of a similar character from his unpublished essays and lectures are now available in the third volume of that edition.[2]

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[1] Cf. in Gödel [1986] the items dated: *1931*(p.181, ftn.48a), *1934*(p.367), *1936*(p.397), and in Gödel [1990] those dated: *1940*(p.97, ftn.20[added 1965]), *1946*(p.151), *1947*(pp.181-183), *1964*(pp.260-261 and 268-270), and *1972a*, Note 2 (pp.305-306).

[2] Cf. in Gödel [1995] the items dated: **1931?*(p.35), **1993o* (p.48), **1951*(pp.306-307), **1961/?*(p.385) and **1970a,b,c*(pp.420-425).

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Given the ready accessibility of these sources, there is n o need for extensive quotation, though several representative passages are singled out below for special attention.

With one possible exception (to be noted in the next section), the single constant that recurs throughout these statements is that the new axioms to be considered are in all cases of a set-theoretic nature. More specifically, to begin with, axioms of higher types, extended into the transfinite, are said to be needed even to settle undecided arithmetical propositions.[3] The first and most succinct statement of this is to be found in the singular footnote 48a of the 1931 incompleteness paper, in which Gödel states that "...the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite...[since] the undecidable propositions constructed here become decidable whenever appropriate higher types are added". In an unpublished lecture from that same period Gödel says that analysis is higher in this sense than number theory and set theory is higher than analysis: "...there are number-theoretic problems that cannot be solved with number-theoretic, but only with analytic or, respectively, set-theoretic methods" (Gödel [1995], p.35). A couple of years later, in his (unpublished) 1933 lecture at a meeting of the Mathematical Association of America in Cambridge, Massachusetts, Gödel said that for the systems S to which his incompleteness theorems apply "...exactly the next higher type not contained in S is necessary to prove this arithmetical proposition...[and moreover] there are arithmetic propositions which cannot be proved even by analysis but only by methods involving extremely large infinite cardinals and similar things" (Gödel [1995], p.48). This assertion of the necessity of axioms of higher type — a.k.a. axioms of infinity in higher set theory — to settle undecided arithmetic (Π^0_1) propositions, is repeated all the way to the final of the references cited here in footnotes 1 and 2 (namely to 1972).

It is only with his famous 1947 article on Cantor's continuum problem that Gödel also pointed to the need for new set-theoretic axioms to settle specifically *set-theoretic* problems, in particular that of the Continuum Hypothesis CH. Of course at that time one only knew through his own work the (relative) consistency of AC and CH with ZF, though Gödel conjectured the falsity of CH and hence its independence from ZFC. Moreover, it was the question of determining the truth value of CH that was to preoccupy him almost exclusively among all set-theoretic problems — except for those which might be ancillary to its solution — for the rest of his life. And rightly so: the continuum problem — to locate $2^{\aleph_0}$ in the scale of the alephs whose existence is forced on us by the well-ordering theorem — is the very first chal-

[3] The kind of proposition in question is sometimes referred to by Gödel as being of "Goldbach type" i.e. in Π^0_1 form, and sometimes as one concerning solutions of Diophantine equations, of the form $(P)D = 0$, where P is a quantifier expression with variables ranging over the natural numbers; cf. more specifically, the lecture notes **193?* in Gödel [1995].

lenging problem of Cantorian set theory, and settling it might be considered to bolster its conceptual coherence. In his 1947 paper, for the decision of CH by new axioms, Gödel mentioned first of all, axioms of infinity:

> The simplest of these ... assert the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $> \aleph_0$. The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes). Other axioms of infinity have been formulated by P. Mahlo. [Very little is known about this section of set theory; but at any rate][4] these axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far. (Gödel [1990], p.182)

However, Gödel goes on to say, quite presciently, that "[a]s for the continuum problem, there is little hope of solving it by means of those axioms of infinity which can be set up on the basis of principles known today...", because his proof of the consistency of CH via the constructible sets model goes through without change when such statements are adjoined as new axioms (indeed there is no hope in this direction if one expects to prove CH false):

> But probably [in the face of this] there exist other [axioms] based on hitherto unknown principles ... which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts. (*ibid.*)

Possible candidates for these were forthcoming through the work of Scott [1961] in which it was shown that the existence of measurable cardinals (MC) implies the negation of the axiom of constructibility, and the later work of Hanf [1964] and of Keisler and Tarski [1964] which showed that measurable cardinals and even weakly compact cardinals must be very much larger than anything obtained by closure conditions on cardinals of the sort leading to hierarchies of inaccessibles. But as we now know through the extensive subsequent work on large cardinals as well as other strong set-theoretic principles such as forms of determinacy, none of those considered at all plausible to date settles CH one way or the other (cf. Martin [1976], Kanamori [1994]). Gödel himself offered only one candidate besides these, in his unpublished 1970 notes containing his "square axioms" concerning so-called scales of functions on the $\aleph_n$'s. The first of these notes (**1970a* in Gödel [1995]) purports to prove that the cardinality of the continuum is $\aleph_2$ while the second (**1970b*, op.cit.) purports to prove that it is $\aleph_1$. However, there are essential gaps in

[4] The section enclosed in brackets was deleted from the 1964 reprinting of the 1947 article (cf. Gödel [1990], p. 260).

both proofs and in any case the axioms considered are far from evident (cf. the introductory note by R.M. Solovay to **1970a,b,c* in Gödel [1995], pp. 405-420).

Gödel's final fall-back position in his 1947 article is to look for axioms which are "so abundant in their verifiable consequences...that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well-established physical theory" (Gödel [1990], p.183). It would take us too far afield to look into the question whether there are any plausible candidates for these. Moreover, there is no space here to consider the arguments given by others in pursuit of the program for new axioms; especially worthy of attention are Maddy [1988, 1988a], Kanamori [1994] and Jensen [1995] among others.

My concern in the rest of this paper is to concentrate on the consideration of axioms which are supposed to be "exactly as evident" as those already accepted. On the face of it this excludes, among others, axioms for "very large" cardinals (compact, measurable, etc.), axioms of determinacy, axioms of randomness, and axioms whose only grounds for accepting them lies in their "fruitfulness" or in their simply having properties analogous to those of $\aleph_0$. Even with this restriction, as we shall see, there is much room for reconsideration of Gödel's program.

2. Where should one look for new axioms?

While the passage to higher types in successive stages, in one form or another, is sufficient to overcome incompleteness with respect to number-theoretic propositions because of the increase in consistency strength at each such stage, it by no means follows that this is the *only* way of adding new axioms in a principled way for that purpose. Indeed, here a quotation from Gödel's remarks in 1946 before the Princeton Bicentennial Conference is very apropos:

> Let us consider, e.g., the concept of demonstrability. It is well known that, in whichever way you make it precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident and justified as those with which you started, and this process of extension can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps; but this does not exclude that all these steps (or at least all of them which give something new for the domain of propositions in which you are interested) could be described and collected together in some non-constructive way. (Gödel [1990], p.151)

It is this passage that I had in mind above as the one possible exception to Gödel's reiterated call for new set-theoretic axioms to settle undecided number-theoretic propositions. It is true that he goes on immediately to say that "[i]n set theory, e.g., the successive extensions can most conveniently be represented by stronger and stronger axioms of infinity". But note that here he is referring to set theory as an *example* of a formalism to which the general idea of expansion by "new axioms exactly as evident and justified as those with which you started" may be applied as a special case. That idea, in the case of formal systems S in the language of arithmetic comes down instead to one form or another of (proof-theoretic) reflection principle, that is a formal scheme to the effect that whatever is provable in S is correct. In its weakest form (assuming the syntax of S effectively and explicitly given), this is the collection of statements

$$(\mathrm{Rfn}_\mathrm{S}) \qquad Prov_\mathrm{S}(\#(A)) \to A$$

for A a closed formula in the language of S, called the *local reflection principle.*[5] This is readily generalized to arbitrary formulas A uniformly in the free variables of A as parameters, in which case it is called the *uniform reflection principle* $\mathrm{RFN_S}$. The axioms $\mathrm{Rfn_S}$, and more generally, $\mathrm{RFN_S}$ may indeed be considered "exactly as evident and justified" as those with which one started. Moreover, as shown by Turing [1939], extension by such axioms may be effectively iterated into the transfinite, in the sense that one can associate with each constructive ordinal notation $a \in O$ a formal system S_a such that the step from any one such system to its successor is described by adjunction of the reflection principle in question, and where all previous adjunctions are simply accumulated at limit s by the formation of their union. These kinds of systematic extensions of a given formal system were called *ordinal logics* by Turing; when I took them up later in 1962, I rechristened them *(transfinite) recursive progressions of axiomatic theories* (cf. Feferman [1962, 1988]). While Turing obtained a completeness result for Π^0_1 statements via the transfinite iteration in this sense of the local reflection principle, and I obtained one for all true arithmetic statements via the iteration of the uniform reflection principle, both completeness results were problematic because they depended crucially on the judicious choice of notations in O, the selection of which was no more "evident and justified" in advance than the statements to be proved.

What was missing in this first attempt to spell out the general idea expressed by Gödel in the above quotation was an explanation of which ordinals — in the constructive sense — ought to be accepted in the iteration procedure. The first modification made to that end (Kreisel [1958], Feferman [1964]) was to restrict to *autonomous* progressions of theories, where one advances to a notation $a \in O$ only if it has been proved in a system S_b, for some b which precedes a, that the ordering specifying a is indeed a well-ordering. It was with this kind of procedure in mind that Kreisel called in his paper [1970] for the study of *all principles of proof and ordinals which are implicit in given concepts.* However, one may question whether it is appropriate at

[5] Note that the consistency statement for S is an immediate consequence of the local reflection principle for S.

all to speak of the concept of ordinal, in whatever way restricted, as being implicit in the concepts of, say, arithmetic. I thus began to pursue a modification of that program in Feferman [1979], where I pr oposed a characterization of that part of mathematical thought which is implicit in our conception of the natural numbers, without any prima-facie use of the notions of ordinal or well-ordering. This turned out to yield a system proof-theoretically equivalent to that proposed as a characterization of *predicativity* in Feferman [1964] and Schütte [1965]. Then in my paper [1991], I proposed more generally, a notion of *reflective closure* of arbitrary schematically axiomatized theories, which gave the same result (proof-theoretically) as the preceding when applied to Peano Arithmetic as initial system. That made use of a partial self-applicative notion of truth, treated axiomatically. The purpose of the present article is to report a new general notion of reflective closure of a quite different form, which I believe is more convincing as an explanation of *everything that one ought to accept if one has accepted given concepts and principles.* In order not to confuse it with the earlier proposal, I shall call this notion that of the *unfolding* of any given schematically formalized system. This will be illustrated here in the case of non-finitist arithmetic as well as the case of set theory. Exact characterizations in more familiar terms have been obtained for the case of non-finitist arithmetic in collaboration with Thomas Strahm; these will be described in Section 4 below. However, there is no space here to give any proofs.

3. How is the unfolding of a system defined?

As we shall see, it is of the essence of the notion of unfolding that we are dealing with schematically presented formal systems. In the usual conception, *formal schemata* for axioms and rules of inference employ *free predicate variables* $P, Q, \ldots$ of various numbers of arguments $n \geq 0$. An appropriate substitution for $P(x_1, \ldots, x_n)$ in such a scheme is a formula $A(x_1, \ldots x_n \ldots)$ which may have additional free variables. (Thus if P is 0-ary, any formula may be substituted for it.) Familiar examples of *axiom schemata* in the propositional and predicate calculi are

$$\neg P \to (P \to Q) \qquad \text{and} \qquad (\forall x)P(x) \to P(t) .$$

Further, in non-finitist arithmetic, we have the *Induction Axiom Scheme*

(IA) $$P(0) \wedge (\forall x)[P(x) \to P(x')] \to (\forall x)P(x) ,$$

while in set theory we have the *Separation* and *Replacement Schemes*

(Sep) $$(\exists b)(\forall x)[x \in b \leftrightarrow x \in a \wedge P(x)], \text{ and}$$

(Repl) $$(\forall x \in a)(\exists! y)P(x, y) \to (\exists b)(\forall y)[y \in b \leftrightarrow (\exists x \in a)P(x, y)] .$$

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Familiar examples of *schematic rules of inference* are, first of all, in the propositional and predicate calculi,

$$P, P \to Q \Rightarrow Q \quad \text{and} \quad [P \to Q(x)] \Rightarrow [P \to (\forall x)Q(x)] \text{ (for } x \text{ not free in } P\text{)},$$

while the scheme for the *Induction Rule* in finitist arithmetic is given by

$$\text{(IR)} \qquad P(0), P(x) \to P(x') \Rightarrow P(x) \ .$$

It is less usual to think of schemata for axioms and rules given by *free function variables* $f, g, \ldots$ But actually, it is more natural to formulate the Replacement Axiom Scheme in functional form as follows:

$$\text{(Repl)}' \qquad (\forall x \in a)(\exists y)[f(x) = y] \to (\exists b)(\forall y)[y \in b \leftrightarrow (\exists x \in a) f(x) = y] \ .$$

Note that here, and for added compelling reasons below, our function variables are treated as ranging over *partial functions.*

The informal philosophy behind the use of schemata here is their *open-endedness.* That is, they are not conceived of as applying to a specific language whose stock of basic symbols is fixed in advance, but rather as applicable to *any* language which one comes to recognize as embodying meaningful basic notions. Put in other terms, *implicit in the acceptance of given schemata is the acceptance of any meaningful substitution instances.* But *which* these instances are need not be determined in advance. Thus, for example, if one accepts the axioms and rules of inference of the classical propositional calculus given in schematic form, one will accept all substitution instances of these schemata in any language which one comes to employ. The same holds for the schemata of the sort given above for arithmetic and set theory. In this spirit, we do not conceive of the function, resp. predicate variables as having a fixed intended range and it is for this reason that th ey are treated as *free* variables. Of course, if one takes it to be meaningful to talk about the totality of partial functions, resp. predicates, of a given domain of objects, then it would be reasonable to bind them too by quantification. In the examples of unfolding given here, it is only in set theory that the issue of whether and to what extent to allow quantification over function variables is unsettled.

Now our question is this: *given a schematic system* S, *which operations and predicates — and which principles concerning them — ought to be accepted if one has accepted* S? The answer for operations is straightforward: *any operation from and to individuals is accepted in the unfolding of* S *which is determined (in successive steps) explicitly or implicitly from the basic operations of* S. Moreover, the *principles* which are added concerning these operations are just those which are derived from the way they are introduced. Ordinarily, we would confine ourselves to the *total operations* obtained in this way, i.e. those which have been proved to be defined for all values of their arguments, but it should not be excluded that their introduction might depend in an essential way on prior *partial operations*, e.g. those introduced by recursive definitions of a general form.

We reformulate the question concerning predicates in operational terms as well, i.e.: *which operations on and to predicates — and which principles concerning them — ought to be accepted if one has accepted* S? For this, it is necessary to tell at the outset *which logical operations on predicates are taken for granted in* S. For example, in the case of non-finitist classical arithmetic these would be (say) the operations $\neg, \wedge$ and $\forall$, while in the case of finitist arithmetic, we would use just $\neg$ and $\wedge$. It proves simplest to treat predicates as propositional functions; thus $\neg$ and $\wedge$ are operations on propositions, while $\forall$ is an operation on functions from individuals to propositions. Now we can add to the operations from individuals to individuals in the unfolding of S also *all those operations from individuals and/or propositions to propositions which are determined explicitly or implicitly (in successive steps) fro m the basic logical operations of* S. Once more, the principles concerning these operations which are included in the expansive closure of S are just those which are derived from the way they are introduced. Finally, *we include in the expansive closure of* S *all the predicates which are generated from the basic predicates of* S *by these operations*; the principles which are taken concerning them are just those that fall out from the principles for the operations just indicated.

This notion of unfolding of a system is spelled out in completely precise terms in the next section for the case of non-finitist arithmetic. But the following two points ought to be noted concerning the general conception described here. First of all, one should not think of the unfolding of a system S as delimiting the range of applicability of the schemata embodied in S. For example, the principle of induction is applicable in every context in which the basic structure of the natural numbers is recognized to be present, even if that context involves concepts and principles not implicit in our basic system for that structure. In particular, it is applicable to impredicative reasoning with sets, even though (as will be shown in the next section) the unfolding closure of arithmetic is limited to predicative reasoning. Secondly, we may expect the language and theorems of the unfolding of (an effectively given system) S to be effectively enumerable, but we should not expect to be a ble to decide which operations introduced by implicit (e.g. recursive fixed-point) definitions are well defined for all arguments, even though it may be just those with which we wish to be concerned in the end. This echoes Gödel's picture of the process of obtaining new axioms which are "just as evident and justified" as those with which we started (quoted in Section 2 above), for which we cannot say in advance exactly what those will be, though we can describe fully the means by which they are to be obtained.

4. The expansive closure of non-finitist arithmetic: what's obtained

Here the starting schematic system NFA (Non-Finitist Arithmetic) has language given by the constant 0, individual variables $x, y, z, \ldots$, the operations

Sc and Pd for successor and predecessor, a free unary predicate variable P and the logical operations $\neg, \wedge$ and $\forall$.

Assuming classical logic, $\wedge, \rightarrow$ and $\exists$ are defined as usual.[6] We write t' for $Sc(t)$ in the following. The axioms of NFA are:

Ax 1. $x' \neq 0$
Ax 2. $Pd(x') = x$
Ax 3. $P(0) \wedge (\forall x)[P(x) \rightarrow P(x')] \rightarrow (\forall x)P(x)$.

Ax 3 is of course our scheme (IA) of induction. Before defining the full unfolding $\mathcal{U}$(NFA) of this system, it is helpful to explain a subsystem $\mathcal{U}_0$(NFA) which might be called the *operational unfolding* of NFA, i.e. where we do not consider which predicates are to be obtained. Basically, the idea is to introduce new operations via a form of generalized recursion theory (g.r.t.) considered axiomatically. The specific g.r.t. referred to is that developed in Moschovakis [1989] and in a different-appearing but equivalent form in Feferman [1991a] and [1996]; both feature *explicit definition* (ED) and *least fixed point recursion* (LFP) and are applicable to arbitrary structures with given functions or functionals of type level ≤ 2 over a given basic domain (or domains). The basic structure to consider in the case of arithmetic is $\langle \mathrm{N}, Sc, Pd, 0 \rangle$, where N is the set of natural numbers. To treat this axiomatically, we simply have to enlarge our language to include the terms for the (in general) partial functions and functionals generated by closure under the schemata for this g.r.t., and add their defining equations as axioms. So we have terms of three types to consider: *individual terms, partial function terms* and *partial functional terms.* The types of these are described as follows, where, to allow for later extension to the case of $\mathcal{U}$(NFA), we posit a set Typ_0 of types of level 0; here we will only need it to contain the type ι of individuals, but below it will be expanded to include the type ι of propositions:

Typ 1. $\iota \in Typ_0$, where ι is the type of individuals. In the following κ, ν range over Typ_0 and $\bar{\iota}$, resp. $\bar{\kappa}$ range over types of finite sequences of individuals, resp. of objects of Typ_0.
Typ 2. τ, σ range over the types of partial functions of the form $\bar{\iota} \stackrel{\sim}{\rightarrow} \nu$, and $\bar{\tau}$ ranges over the types of finite sequences of such.
Typ 3. $(\bar{\tau}, \bar{\kappa} \stackrel{\sim}{\rightarrow} \nu)$ is used as types of partial functionals.

Note that objects of partial function type take only individuals as arguments; this is to insure that propositional functions, to be considered below, are just such functions. On the other hand, we may have partial functionals of type described under Typ 3 in which the sequence $\bar{\tau}$ is empty, and these reduce to partial functions of any objects of basic type in Typ_0.

[6] All our notions and results carry over directly to NFA treated in intuitionistic logic; the only difference in that case is that we take the full list of logical operations, $\neg, \wedge, \vee, \rightarrow, \forall$, and $\exists$ as basic.

The terms $r, s, t, u, \ldots$ of the various types under Typ 1 – Typ 3 are generated as follows, where we use $r : \rho$ to indicate that the term r is of type ρ.

Tm 1. For each $\kappa \in Typ_0$, we have infinitely many variables $x, y, z, \ldots$ of type κ.
Tm 2. $0 : \iota$.
Tm 3. $Sc(t) : \iota$ and $Pd(t) : \iota$ for $t : \iota$.
Tm 4. For each τ we have infinitely many partial function variables $f, g, h, \ldots$ of type τ.
Tm 5. $Cond(s, t, u, v) : (\bar{\tau}, \bar{\kappa}, \iota, \iota \tilde{\rightarrow} \nu)$ for $s, t : (\bar{\tau}, \bar{\kappa} \tilde{\rightarrow} \nu)$ and $u, v : \iota$.
Tm 6. $s(\bar{t}, \bar{u}) : \nu$ for $s : (\bar{\tau}, \bar{\kappa} \tilde{\rightarrow} \nu)$, $\bar{t} : \bar{\tau}$, $\bar{u} : \bar{\kappa}$.
Tm 7. $\lambda \bar{f}, \bar{x}.t : (\bar{\tau}, \bar{\kappa} \tilde{\rightarrow} \nu)$ for $\bar{f} : \bar{\tau}, \bar{x} : \bar{\kappa}, t : \nu$.
Tm 8. LFP $(\lambda f, \bar{x}.t) : (\bar{\iota} \tilde{\rightarrow} \nu)$ for $f : \bar{\iota} \tilde{\rightarrow} \nu, \bar{x} : \bar{\iota}, \; t : \nu$.

We now specialize this system of types and terms to just what is needed for $\mathcal{U}_0$(NFA), by taking $Typ_0 = \{\iota\}$. The formulas $A, B, C, \ldots$ of $\mathcal{U}_0$(NFA) are then generated as follows:

Fm 1. The atomic formulas are $s = t, s\downarrow$, and $P(s)$ for $s, t : \iota$.
Fm 2. If A, B are formulas then so also are $\neg A, A \wedge B$, and $\forall x A$.

As indicated above, formulas $A \vee B, A \rightarrow B$, and $\exists x A$ are defined as usual in classical logic. We write $s \simeq t$ for $[s\downarrow \vee t\downarrow \rightarrow s = t]$. Below we write $t\ [\bar{f}, \bar{x}]$, resp. $A\ [\bar{f}, \bar{x}]$ for a term, resp. formula, with designated sequences of free variables $\bar{f}, \bar{x}$; it is not excluded that t, resp. A may contain other free variables when using this notation. Since we are dealing with possibly undefined (individual) terms t, the underlying system of logic to be used is the *logic of partial terms* (LPT) introduced by Beeson [1985], pp. 97-99, where $t\downarrow$ is read as: t is defined. Briefly, the changes to be made from usual predicate logic are, first, that the axiom for $\forall$-instantiation is modified to

$$\forall x A(x) \wedge t\downarrow \rightarrow A(t) .$$

In addition, it is assumed that $\forall x(x\downarrow)$, i.e. only compound terms may fail to be defined (or put otherwise, non-existent individuals are not countenanced in LPT). It is further assumed that if a compound term is defined then all its subterms are defined ("strictness" axioms). Finally, one assumes that if $s = t$ holds then both s, t are defined and if $P(s)$ holds then s is defined. Note that $(s\downarrow) \leftrightarrow \exists x(s = x)$, so definedness need not be taken as a basic symbol.

The axioms of $\mathcal{U}_0$(NFA) follow the obvious intended meaning of the new compound terms introduced by the clauses Tm 5-8:

Ax 4. $(Cond(s, t, u, u))(\bar{f}, \bar{x}) \simeq s(\bar{f}, \bar{x}) \wedge [u \neq v \rightarrow (Cond(s, t, u, v))(\bar{f}, \bar{x}) \simeq t(\bar{f}, \bar{x})]$.
Ax 5. $(\lambda \bar{f}, \bar{x}.s[\bar{f}, \bar{x}])(\bar{t}, \bar{u}) \simeq s[\bar{t}, \bar{u}]$.
Ax 6. For $\varphi = \mathrm{LFP}(\lambda f, \bar{x}.t[f, \bar{x}])$, we have:
(i) $\varphi(\bar{x}) \simeq t[\varphi, \bar{x}]$