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Excerpt
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1 Vector and Tensor Calculus

1.1 Introduction

Before we can start with biomechanics, it is necessary to introduce some basic mathematical concepts and to introduce the mathematical notation that will be used throughout the book. The present chapter is aimed at understanding some of the basics of vector calculus, which are necessary to elucidate the concepts of force and momentum that will be treated in the next chapter.

1.2 Definition of a Vector

A **vector** is a mathematical entity having both a magnitude (length or size) and a direction. For a vector $\vec{a}$, it holds (see Fig. 1.1), that:

$$\vec{a} = a\vec{e}. \tag{1.1}$$

The **length** of the vector $\vec{a}$ is denoted by $|\vec{a}|$ and is equal to the length of the arrow. The length is equal to a, when a is positive, and equal to $-a$ when a is negative. The **direction** of $\vec{a}$ is given by the unit vector $\vec{e}$ combined with the sign of a. The unit vector $\vec{e}$ has length 1. The vector $\vec{0}$ has length zero.

1.3 Vector Operations

Multiplication of a vector $\vec{a} = a\vec{e}$ by a positive scalar α yields a vector $\vec{b}$ having the same direction as $\vec{a}$ but a different magnitude $\alpha|\vec{a}|$:

$$\vec{b} = \alpha\vec{a} = \alpha a\vec{e}. \tag{1.2}$$

This makes sense: pulling twice as hard on a wire creates a force in the wire having the same orientation (the direction of the wire does not change), but with a magnitude that is twice as large.

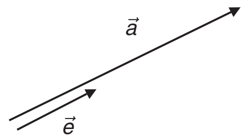

Figure 1.1

The vector $\vec{a} = a\vec{e}$ with $a > 0$.

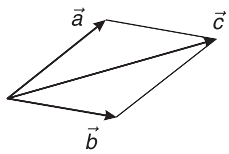

Figure 1.2

Graphical representation of the sum of two vectors: $\vec{c} = \vec{a} + \vec{b}$.

The **sum** of two vectors $\vec{a}$ and $\vec{b}$ is a new vector $\vec{c}$, equal to the diagonal of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ (see Fig. 1.2):

$$\vec{c} = \vec{a} + \vec{b}. \tag{1.3}$$

This may be interpreted as follows. Imagine two thin wires which are attached to a point P. The wires are being pulled at in two different directions according to the vectors $\vec{a}$ and $\vec{b}$. The length of each vector represents the magnitude of the pulling force. The net force vector exerted on the attachment point P is the vector sum of the two vectors $\vec{a}$ and $\vec{b}$. If the wires are aligned with each other and the pulling direction is the same, the resulting force direction clearly coincides with the direction of the two wires, and the length of the resulting force vector is the sum of the two pulling forces. Alternatively, if the two wires are aligned but the pulling forces are in opposite directions and of equal magnitude, the resulting force exerted on point P is the zero vector $\vec{0}$.

The **inner product** or **dot product** of two vectors is a scalar quantity, defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\phi), \tag{1.4}$$

where ϕ is the smallest angle between $\vec{a}$ and $\vec{b}$ (see Fig. 1.3). The inner product is **commutative**, i.e.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}. \tag{1.5}$$

The inner product can be used to define the length of a vector, since the inner product of a vector with itself yields ($\phi = 0$):

$$\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| \cos(0) = |\vec{a}|^2. \tag{1.6}$$

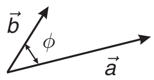

Figure 1.3

Definition of the angle ϕ.

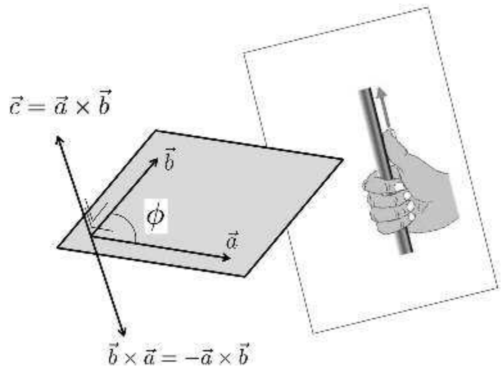

Figure 1.4

Vector product $\vec{c} = \vec{a} \times \vec{b}$. The direction of vector $\vec{c}$ is determined by the corkscrew or right-hand rule.

If two vectors are perpendicular to each other the inner product of these two vectors is equal to zero, since in that case $\phi = \frac{\pi}{2}$:

$$\vec{a} \cdot \vec{b} = 0, \text{ if } \phi = \frac{\pi}{2}. \tag{1.7}$$

The **cross product** or **vector product** of two vectors $\vec{a}$ and $\vec{b}$ yields a new vector $\vec{c}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed system. The vector $\vec{c}$ is denoted as

$$\vec{c} = \vec{a} \times \vec{b}\,. \tag{1.8}$$

The length of the vector $\vec{c}$ is given by

$$|\vec{c}| = |\vec{a}||\vec{b}| \sin(\phi), \tag{1.9}$$

where ϕ is the smallest angle between $\vec{a}$ and $\vec{b}$. The length of $\vec{c}$ equals the area of the parallelogram spanned by the vectors $\vec{a}$ and $\vec{b}$. The vector system $\vec{a}$, $\vec{b}$ and $\vec{c}$ forms a right-handed system, meaning that if a corkscrew were used rotating from $\vec{a}$ to $\vec{b}$ the corkscrew would move into the direction of $\vec{c}$ (see Fig. 1.4).

The vector product of a vector $\vec{a}$ with itself yields the zero vector, since in that case $\phi = 0$:

$$\vec{a} \times \vec{a} = \vec{0}. \tag{1.10}$$

The vector product is **not** commutative, since the vector product of $\vec{b}$ and $\vec{a}$ yields a vector that has the opposite direction to the vector product of $\vec{a}$ and $\vec{b}$:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}. \tag{1.11}$$

The **triple product** of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is a scalar, defined by

$$\vec{a} \times \vec{b} \cdot \vec{c} = (\vec{a} \times \vec{b}) \cdot \vec{c}. \tag{1.12}$$

So, first the vector product of $\vec{a}$ and $\vec{b}$ is determined and subsequently the inner product of the resulting vector with the third vector $\vec{c}$ is taken. If all three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors, while the triple product is equal to zero, then the vector $\vec{c}$ lies in the plane spanned by the vectors $\vec{a}$ and $\vec{b}$. This can be explained by the fact that the vector product of $\vec{a}$ and $\vec{b}$ yields a vector perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$. Conversely, this implies that if the triple product is non-zero then the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not in the same plane. In that case the absolute value of the triple product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ equals the volume of the parallelepiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c}$.

The **dyadic** or **tensor product** of two vectors $\vec{a}$ and $\vec{b}$ defines a linear transformation operator called a **dyad** $\vec{a}\vec{b}$. Application of a dyad $\vec{a}\vec{b}$ to a vector $\vec{p}$ yields a vector into the direction of $\vec{a}$, where $\vec{a}$ is multiplied by the inner product of $\vec{b}$ and $\vec{p}$:

$$\vec{a}\vec{b} \cdot \vec{p} = \vec{a}\,(\vec{b} \cdot \vec{p}). \tag{1.13}$$

So, application of a dyad to a vector transforms this vector into another vector. This transformation is linear, as can be seen from

$$\vec{a}\vec{b} \cdot (\alpha\vec{p} + \beta\vec{q}) = \vec{a}\vec{b} \cdot \alpha\vec{p} + \vec{a}\vec{b} \cdot \beta\vec{q} = \alpha\vec{a}\vec{b} \cdot \vec{p} + \beta\vec{a}\vec{b} \cdot \vec{q}. \tag{1.14}$$

The transpose of a dyad $(\vec{a}\vec{b})^{\mathrm{T}}$ is defined by

$$(\vec{a}\vec{b})^{\mathrm{T}} \cdot \vec{p} = \vec{b}\vec{a} \cdot \vec{p}, \tag{1.15}$$

or simply

$$(\vec{a}\vec{b})^{\mathrm{T}} = \vec{b}\vec{a}. \tag{1.16}$$

An operator $\boldsymbol{A}$ that transforms a vector $\vec{a}$ into another vector $\vec{b}$ according to

$$\vec{b} = \boldsymbol{A} \cdot \vec{a}, \tag{1.17}$$

is called a second-order tensor $\boldsymbol{A}$. This implies that the dyadic product of two vectors is a second-order tensor.

In three-dimensional space, a set of three vectors $\vec{c}_1$, $\vec{c}_2$ and $\vec{c}_3$ is called a **basis** if the triple product of the three vectors is non-zero, hence if all three vectors are non-zero vectors and if they do not lie in the same plane:

$$\vec{c}_1 \times \vec{c}_2 \cdot \vec{c}_3 \neq 0. \tag{1.18}$$

The three vectors $\vec{c}_1$, $\vec{c}_2$ and $\vec{c}_3$ composing the basis are called basis vectors.

If the basis vectors are mutually perpendicular vectors, the basis is called an **orthogonal** basis. If such basis vectors have unit length, then the basis is called **orthonormal**. A **Cartesian basis** is an orthonormal, right-handed basis with basis vectors independent of the location in the three-dimensional space. In the following we will indicate the Cartesian basis vectors with $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$.

1.4 Decomposition of a Vector with Respect to a Basis

As stated above, a Cartesian vector basis is an orthonormal basis. Any vector can be decomposed into the sum of, at most, three vectors parallel to the three basis vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$:

$$\vec{a} = a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z. \tag{1.19}$$

The components a_x, a_y and a_z can be found by taking the inner product of the vector $\vec{a}$ with respect to each of the basis vectors:

$$\begin{aligned} a_x &= \vec{a} \cdot \vec{e}_x \\ a_y &= \vec{a} \cdot \vec{e}_y \\ a_z &= \vec{a} \cdot \vec{e}_z, \end{aligned} \tag{1.20}$$

where use is made of the fact that the basis vectors have unit length and are mutually orthogonal, for example:

$$\vec{a} \cdot \vec{e}_x = a_x\vec{e}_x \cdot \vec{e}_x + a_y\vec{e}_y \cdot \vec{e}_x + a_z\vec{e}_z \cdot \vec{e}_x = a_x. \tag{1.21}$$

The components, say a_x, a_y and a_z, of a vector $\vec{a}$ with respect to the Cartesian vector basis, may be collected in a **column**, denoted by $\underset{\sim}{a}$:

$$\underset{\sim}{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}. \tag{1.22}$$

So, with respect to a Cartesian vector basis, any vector $\vec{a}$ may be decomposed into components that can be collected in a column:

$$\vec{a} \longleftrightarrow \underset{\sim}{a}\,. \tag{1.23}$$

This ‘transformation’ is only possible and meaningful if the vector basis with which the components of the column $\underset{\sim}{a}$ are defined has been specified. The choice of a different vector basis leads to a different column representation $\underset{\sim}{a}$ of the vector

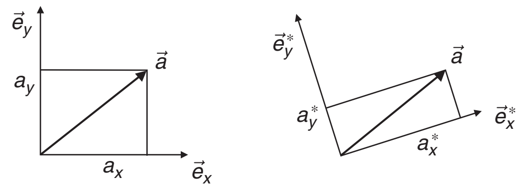

Figure 1.5

Vector $\vec{a}$ with respect to vector bases $\{\vec{e}_x, \vec{e}_y\}$ and $\{\vec{e}_x^*, \vec{e}_y^*\}$.

$\vec{a}$, as illustrated in Fig. 1.5. The vector $\vec{a}$ has two different column representations, $\underset{\sim}{a}$ and $\underset{\sim}{a}^*$, depending on which vector basis is used. If, in a two-dimensional context, $\{\vec{e}_x, \vec{e}_y\}$ is used as a vector basis then

$$\vec{a} \longleftrightarrow \underset{\sim}{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}, \tag{1.24}$$

while, if $\{\vec{e}_x^{\,*}, \vec{e}_y^{\,*}\}$ is used as a vector basis:

$$\vec{a} \longleftrightarrow \underset{\sim}{a}^* = \begin{bmatrix} a_x^* \\ a_y^* \end{bmatrix}. \tag{1.25}$$

Consequently, with respect to a Cartesian vector basis, vector operations such as multiplication, addition, inner product and dyadic product may be rewritten as 'column' (actually matrix) operations.

Multiplication of a vector $\vec{a} = a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z$ with a scalar α yields a new vector, say $\vec{b}$:

$$\begin{aligned}\vec{b} = \alpha\vec{a} &= \alpha(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\ &= \alpha a_x\vec{e}_x + \alpha a_y\vec{e}_y + \alpha a_z\vec{e}_z.\end{aligned} \tag{1.26}$$

So

$$\vec{b} = \alpha\vec{a} \longleftrightarrow \underset{\sim}{b} = \alpha\underset{\sim}{a}. \tag{1.27}$$

The sum of two vectors $\vec{a}$ and $\vec{b}$ leads to

$$\vec{c} = \vec{a} + \vec{b} \longleftrightarrow \underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b}. \tag{1.28}$$

Using the fact that the Cartesian basis vectors have unit length and are mutually orthogonal, the inner product of two vectors $\vec{a}$ and $\vec{b}$ yields a scalar c according to

$$\begin{aligned}c = \vec{a}\cdot\vec{b} &= (a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z)\cdot(b_x\vec{e}_x + b_y\vec{e}_y + b_z\vec{e}_z) \\ &= a_x b_x + a_y b_y + a_z b_z.\end{aligned} \tag{1.29}$$

In column notation this result is obtained via

$$c = \underset{\sim}{a}^{\mathrm{T}}\underset{\sim}{b}\,, \tag{1.30}$$

where $\underset{\sim}{a}^{\mathrm{T}}$ denotes the **transpose** of the column $\underset{\sim}{a}$, defined as

$$\underset{\sim}{a}^{\mathrm{T}} = \left[a_x \; a_y \; a_z\right], \tag{1.31}$$

such that:

$$\underset{\sim}{a}^{\mathrm{T}} \underset{\sim}{b} = \left[a_x \; a_y \; a_z\right] \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z. \tag{1.32}$$

Using the properties of the basis vectors of the Cartesian vector basis:

$$\begin{aligned} \vec{e}_x \times \vec{e}_x &= \vec{0} \\ \vec{e}_x \times \vec{e}_y &= \vec{e}_z \\ \vec{e}_x \times \vec{e}_z &= -\vec{e}_y \\ \vec{e}_y \times \vec{e}_x &= -\vec{e}_z \\ \vec{e}_y \times \vec{e}_y &= \vec{0} \\ \vec{e}_y \times \vec{e}_z &= \vec{e}_x \\ \vec{e}_z \times \vec{e}_x &= \vec{e}_y \\ \vec{e}_z \times \vec{e}_y &= -\vec{e}_x \\ \vec{e}_z \times \vec{e}_z &= \vec{0}, \end{aligned} \tag{1.33}$$

the vector product of a vector $\vec{a}$ and a vector $\vec{b}$ is directly computed by means of

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \times (b_x\vec{e}_x + b_y\vec{e}_y + b_z\vec{e}_z) \\ &= (a_y\, b_z - a_z\, b_y)\vec{e}_x + (a_z\, b_x - a_x\, b_z)\vec{e}_y + (a_x\, b_y - a_y\, b_x)\vec{e}_z. \end{aligned} \tag{1.34}$$

If by definition $\vec{c} = \vec{a} \times \vec{b}$, then the associated column $\underset{\sim}{c}$ can be written as:

$$\underset{\sim}{c} = \begin{bmatrix} a_y\, b_z - a_z\, b_y \\ a_z\, b_x - a_x\, b_z \\ a_x\, b_y - a_y\, b_x \end{bmatrix}. \tag{1.35}$$

The dyadic product $\vec{a}\vec{b}$ transforms another vector $\vec{c}$ into a vector $\vec{d}$, according to the definition

$$\vec{d} = \vec{a}\vec{b} \cdot \vec{c} = \boldsymbol{A} \cdot \vec{c}\,, \tag{1.36}$$

with $\boldsymbol{A}$ the second-order tensor equal to the dyadic product $\vec{a}\vec{b}$. In column notation this is equivalent to

$$\underset{\sim}{d} = \underset{\sim}{a}\,(\underset{\sim}{b}^{\mathrm{T}}\underset{\sim}{c}\,) \;=\; (\underset{\sim}{a}\,\underset{\sim}{b}^{\mathrm{T}})\underset{\sim}{c}\,, \tag{1.37}$$

with $\underset{\sim}{a}\,\underset{\sim}{b}^{\mathrm{T}}$ a (3×3) matrix given by

$$\underline{A} = \underset{\sim}{a}\,\underset{\sim}{b}^{\mathrm{T}} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}, \tag{1.38}$$

or

$$\underset{\sim}{d} = \underline{A}\,\underset{\sim}{c}. \tag{1.39}$$

In this case $\underline{A}$ is called the matrix representation of the second-order tensor $\boldsymbol{A}$, as the comparison of Eqs. (1.36) and (1.39) reveals.

Example 1.1 Suppose we can write the vectors $\vec{a}$ and $\vec{b}$ as the following linear combination of the Cartesian basis vectors $\vec{e}_x$ and $\vec{e}_y$:

$$\vec{a} = \vec{e}_x + 2\vec{e}_y$$
$$\vec{b} = 2\vec{e}_x + 5\vec{e}_y,$$

and we wish to determine the inner and vector product of both vectors. Then:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (\vec{e}_x + 2\vec{e}_y) \cdot (2\vec{e}_x + 5\vec{e}_y) \\ &= 2\vec{e}_x \cdot \vec{e}_x + 5\vec{e}_x \cdot \vec{e}_y + 4\vec{e}_y \cdot \vec{e}_x + 10\vec{e}_y \cdot \vec{e}_y \\ &= 2 + 10 \\ &= 12. \end{aligned}$$

When using column notation we can also write:

$$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \underset{\sim}{b} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix},$$

and:

$$\vec{a} \cdot \vec{b} = \underset{\sim}{a}^T \underset{\sim}{b} = [1 \quad 2 \quad 0] \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = 12.$$

For the vector product a similar procedure can be used:

$$\begin{aligned} \vec{a} \times \vec{b} &= (\vec{e}_x + 2\vec{e}_y) \times (2\vec{e}_x + 5\vec{e}_y) \\ &= 2\vec{e}_x \times \vec{e}_x + 5\vec{e}_x \times \vec{e}_y + 4\vec{e}_y \times \vec{e}_x + 10\vec{e}_y \times \vec{e}_y \\ &= \vec{0} + 5\vec{e}_z - 4\vec{e}_z + \vec{0} \\ &= \vec{e}_z. \end{aligned}$$

When using column notation for the vector product, Eq. (1.35) has to be used.

Example 1.2 Consider the Cartesian basis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$. We want to know whether the following three vectors given by:

$$\vec{\varepsilon}_1 = 2\vec{e}_x$$
$$\vec{\varepsilon}_2 = \vec{e}_x + 2\vec{e}_y$$
$$\vec{\varepsilon}_3 = \vec{e}_y + 3\vec{e}_z$$

could also be used as a basis. For this purpose we have to determine whether the vectors are independent, and consequently we calculate the triple product:

$$(\vec{\varepsilon}_1 \times \vec{\varepsilon}_2) \cdot \vec{\varepsilon}_3 = \left[2\vec{e}_x \times (\vec{e}_x + 2\vec{e}_y)\right] \cdot (\vec{e}_y + 3\vec{e}_z)$$
$$= 4\vec{e}_z \cdot (\vec{e}_y + 3\vec{e}_z) = 12 \neq 0.$$

This means that the three vectors are independent and might be used as a basis. However, they are not perpendicular and do not have length 1.

Example 1.3 With respect to a Cartesian basis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ the following vectors are defined:

$$\vec{a} = \vec{e}_x + 2\vec{e}_y$$
$$\vec{b} = 2\vec{e}_x + 5\vec{e}_y$$
$$\vec{c} = 3\vec{e}_x.$$

We want to determine the dyadic products $\boldsymbol{A} = \vec{a}\vec{b}$, $\boldsymbol{A}^T = \vec{b}\vec{a}$ and the result of $\boldsymbol{A} \cdot \vec{c}$ and $\boldsymbol{A}^T \cdot \vec{c}$. Write

$$\boldsymbol{A} = \vec{a}\vec{b} = (\vec{e}_x + 2\vec{e}_y)(2\vec{e}_x + 5\vec{e}_y)$$
$$= 2\vec{e}_x\vec{e}_x + 5\vec{e}_x\vec{e}_y + 4\vec{e}_y\vec{e}_x + 10\vec{e}_y\vec{e}_y,$$

$$\boldsymbol{A}^T = \vec{b}\vec{a} = (2\vec{e}_x + 5\vec{e}_y)(\vec{e}_x + 2\vec{e}_y)$$
$$= 2\vec{e}_x\vec{e}_x + 4\vec{e}_x\vec{e}_y + 5\vec{e}_y\vec{e}_x + 10\vec{e}_y\vec{e}_y$$

and:

$$\boldsymbol{A} \cdot \vec{c} = \vec{a}\vec{b} \cdot \vec{c} = (2\vec{e}_x\vec{e}_x + 5\vec{e}_x\vec{e}_y + 4\vec{e}_y\vec{e}_x + 10\vec{e}_y\vec{e}_y) \cdot 3\vec{e}_x$$
$$= 6\vec{e}_x + 12\vec{e}_y,$$

$$\boldsymbol{A}^T \cdot \vec{c} = \vec{b}\vec{a} \cdot \vec{c} = (2\vec{e}_x\vec{e}_x + 4\vec{e}_x\vec{e}_y + 5\vec{e}_y\vec{e}_x + 10\vec{e}_y\vec{e}_y) \cdot 3\vec{e}_x$$
$$= 6\vec{e}_x + 15\vec{e}_y.$$

We can also use matrix notation. In that case $\underline{A}$ is the matrix representation of $\boldsymbol{A}$.

$$\underline{A} = \underset{\sim}{a}\underset{\sim}{b}^T = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} [2 \;\; 5 \;\; 0] = \begin{bmatrix} 2 & 5 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and:

$$\underset{\sim}{A}\,\underset{\sim}{c} = \begin{bmatrix} 2 & 5 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 0 \end{bmatrix},$$

$$\underset{\sim}{A}^T\,\underset{\sim}{c} = \begin{bmatrix} 2 & 4 & 0 \\ 5 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 0 \end{bmatrix}.$$

which clearly is not the same as: $\underset{\sim}{A}\,\underset{\sim}{c}$.

1.5 Some Mathematical Preliminaries on Second-Order Tensors

In this section we will elaborate a bit more about second-order tensors, because they play an important role in continuum mechanics. Remember that every dyadic product of two vectors is a second-order tensor and every second-order tensor can be written as the sum of dyadic products.

An arbitrary second-order tensor $\boldsymbol{M}$ can be written with respect to the Cartesian basis introduced earlier as:

$$\begin{aligned} \boldsymbol{M} = {} & M_{xx}\vec{e}_x\vec{e}_x + M_{xy}\vec{e}_x\vec{e}_y + M_{xz}\vec{e}_x\vec{e}_z \\ & + M_{yx}\vec{e}_y\vec{e}_x + M_{yy}\vec{e}_y\vec{e}_y + M_{yz}\vec{e}_y\vec{e}_z \\ & + M_{zx}\vec{e}_z\vec{e}_x + M_{zy}\vec{e}_z\vec{e}_y + M_{zz}\vec{e}_z\vec{e}_z. \end{aligned} \tag{1.40}$$

The components of the tensor $\boldsymbol{M}$ are stored in the associated matrix $\underset{\sim}{M}$ defined as

$$\underset{\sim}{M} = \begin{bmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{bmatrix}. \tag{1.41}$$

A tensor identifies a linear transformation. If the vector $\vec{b}$ is the result of the tensor $\boldsymbol{M}$ operating on vector $\vec{a}$, this is written as: $\vec{b} = \boldsymbol{M} \cdot \vec{a}$. In component form, this leads to:

$$\begin{aligned} \vec{b} = {} & (M_{xx}\vec{e}_x\vec{e}_x + M_{xy}\vec{e}_x\vec{e}_y + M_{xz}\vec{e}_x\vec{e}_z \\ & + M_{yx}\vec{e}_y\vec{e}_x + M_{yy}\vec{e}_y\vec{e}_y + M_{yz}\vec{e}_y\vec{e}_z \\ & + M_{zx}\vec{e}_z\vec{e}_x + M_{zy}\vec{e}_z\vec{e}_y + M_{zz}\vec{e}_z\vec{e}_z) \cdot (a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \end{aligned}$$