TOWARDS HIGHER MATHEMATICS: A COMPANION

Containing a large and varied set of problems, this rich resource will allow students to stretch their mathematical abilities beyond the school syllabus, and bridge the gap to university-level mathematics. Many proofs are provided to better equip students for the transition to university.

The author covers substantial extension material using the language of sixth form mathematics, thus enabling students to understand the more complex material. Exercises are carefully chosen to introduce students to some central ideas, without building up large amounts of abstract technology. There are over 1500 carefully graded exercises, with hints included in the text, and solutions available online. Historical and contextual asides highlight each area of mathematics and show how it has developed over time.

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In memory of Jazz.

Never the most focused of students, but your tender presence in tutorials
and around college is sorely missed.

Contents

Glossary

Relating to Sets

The set of **real numbers** is denoted as $\mathbb{R}$.

A **natural number** is a non-negative whole number. The set of natural numbers is denoted $\mathbb{N}$. Note that by this definition 0 is a natural number; some texts consider only positive whole numbers to be natural.

An **integer** is a whole number. The set of integers is denoted $\mathbb{Z}$. The letter zed is used from the German word *Zahlen* meaning 'numbers'.

A **rational number** is a real number which can be written as the ratio of two integers. The set of rational numbers is denoted $\mathbb{Q}$. A real number which is not rational is said to be **irrational**.

If S is a set, we write $x \in S$ to denote that x is an **element** of S. So $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$, as $\sqrt{2}$ is real but irrational.

If S is a set, the notation $\{x \in S : P(x)\}$ denotes the **subset** of S consisting of those elements x that satisfy some property P. So, for example, $\{x \in \mathbb{R} : x > 0\}$ denotes the set of positive real numbers.

Two sets A and B are said to be **disjoint** if there is no element common to both sets. Equivalently this means that the intersection $A \cap B$ is empty.

A **partition** of a set X is a collection of disjoint non-empty subsets of X whose union equals X. So every element of X is an element of one and only one subset from the partition.

Relating to Logic

Given statements P and Q, we say that P **implies** Q, and write $P \Longrightarrow Q$, if whenever P is true then Q is true. The **converse** implication is $Q \Longrightarrow P$. So, for example, the statement '$x > 0$' implies '$x^2 > 0$'. In this case the converse is false: '$x^2 > 0$' does not imply '$x > 0$', as $(-1)^2 > 0$.

If statement P implies statement Q, we say that Q is **necessary** for P and P is **sufficient** for Q. For example, '$x = 2$' implies 'x is even'. It is necessary for x to be even for x to equal 2; it is sufficient for x to equal 2 for x to be even. It is not necessary for x to equal 2 for x to be even, and it is not sufficient for x to be even for x to equal 2; these last two facts show the converse is false: 'x is even' does not imply '$x = 2$'.

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More Information

Given two statements P and Q such that $P \Longrightarrow Q$ and $Q \Longrightarrow P$, then we say P is true **if and only if** Q is true and write $P \Longleftrightarrow Q$. We also say that P is **necessary and sufficient** for Q.

Relating to Functions

Given sets X and Y, a **function** or **map** f from X to Y is a rule which assigns an element $f(x)$ of Y to each x in X. This is commonly denoted as $f\colon X \to Y$.

Given a function f from X to Y, we refer to X as the **domain** and Y as the **codomain**. The **image** or **range** of f is the subset $\{f(x) : x \in X\}$ of Y. So given the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$, its image is the set of non-negative real numbers. A **bijection** f from a set S to a set T is a function satisfying:

(i) if $f(s_1) = f(s_2)$ where s_1, s_2 are in S, then $s_1 = s_2$; that is, f is **injective** or **1-1**.
(ii) if t is in T then there is s in S such that $f(s) = t$; that is, f is **surjective** or **onto**.

For example, as functions from $\mathbb{R}$ to $\mathbb{R}$, $2x+1$ is bijective, e^x is only 1–1, $x^3 - x$ is only onto and $\sin x$ is neither.

A function f from a set S to a set T is said to be **invertible** if there is a map g from T to S such that $g(f(s)) = s$ for all s in S and $f(g(t)) = t$ for all t in T. We refer to g as the **inverse** of f and write $g = f^{-1}$. It is a fact that a function is invertible if and only if it is bijective.

Given a function f from a set S to a set T, and if R is a subset of S, then the **restriction** of f to R, denoted $f|_R$, is the map from R to T defined by $f|_R(r) = f(r)$ for all r in R.

Let R, S, T be sets and f a map from R to S and g a map from S to T. Then the **composition** $g \circ f$ from R to T is defined as

$$(g \circ f)(r) = g(f(r)) \qquad \text{for all } r \text{ in } R.$$

Let f be a real-valued function, defined on some subset S of $\mathbb{R}$. We say f is **increasing** if $f(x) \leqslant f(y)$ whenever $x \leqslant y$ and is **strictly increasing** if $f(x) < f(y)$ whenever $x < y$. Similarly we say f is **decreasing** if $f(x) \geqslant f(y)$ whenever $x \leqslant y$ and is **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$.

Let f be a real-valued function, defined on some subset S of $\mathbb{R}$. We say that f is **bounded** if there is a positive number M such that $|f(x)| < M$ for all x in S. A function which is not bounded is said to be **unbounded**. For example, $\sin x$ and e^{-x^2} are bounded on $\mathbb{R}$, whilst x^{-1} is unbounded on the interval $0 < x < 1$.

A **sequence** (x_n) is an infinite ordered list of real (or complex) numbers. We denote the nth term of the sequence as x_n. Sequences typically begin from $n = 0$ or $n = 1$.

A real sequence (x_n) is **increasing** if $x_m \leqslant x_n$ when $m \leqslant n$ and **strictly increasing** if $x_m < x_n$ when $m < n$. Likewise (x_n) is **decreasing** if $x_m \geqslant x_n$ when $m \leqslant n$ and **strictly decreasing** if $x_m > x_n$ when $m < n$.

A real (or complex) sequence (x_n) is said to be **bounded** if there is a positive number M such that $|x_n| < M$ for all n and is otherwise said to be **unbounded**. For example $\left(2^{-n}\right)$ is bounded while $\left(n^2\right)$ is unbounded.

Miscellaneous

An **equivalence relation** on a set S is a binary relation $\sim$ on S, binary here meaning $\sim$ compares two elements of S and says whether the relation is true or not for that pair. Further $\sim$ must satisfy (i) $s \sim s$ for all s in S, (ii) if $s \sim t$ then $t \sim s$, (iii) if $s \sim t$ and $t \sim u$ then $s \sim u$. So similarity is an equivalence relation on the set of triangles in the xy-plane. However, $\leqslant$ is not an equivalence relation on $\mathbb{R}$, as it does not satisfy condition (ii); for example, '$3 \leqslant 4$' is true but '$4 \leqslant 3$' is false.

A **group** $(G,*)$ is a set G together with a binary operation $*$ on G, binary here meaning that $*$ takes two inputs g_1, g_2 from G and returns an output $g_1 * g_2$ in G. Further:

(i) the operation needs to be **associative**, that is, $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all g_1, g_2, g_3 in G;
(ii) there needs to be an identity element e satisfying $e * g = g = g * e$ for all g;
(iii) for each element there needs to be an inverse g^{-1} such that $g * g^{-1} = e = g^{-1} * g$.

There are many examples of groups in mathematics including the real numbers under addition, the non-zero rational numbers under multiplication, and the bijections from a set to itself under composition.

Two positive integers m, n are said to be **coprime** if the only positive integer which divides both is 1.

Given a natural number n, its **factorial** $n!$ is defined inductively by $0! = 1$ and $n! = n \times (n-1)!$ for $n \geqslant 1$.

Two random variables X and Y are said to be **independent** if for all x, y we have

$$P(X = x \text{ and } Y = y) = P(X = x) \times P(Y = y).$$

For example, two rolls of a fair die are independent, but the sum of the rolls is not independent of the first roll.

Symbols

General Notation

Notation	Meaning	Page	Notation	Meaning	Page
$\mathbb{R}$	the real numbers	p.ix	$\binom{n}{k}$	binomial coefficient	p.85
$\mathbb{Q}$	the rational numbers	p.ix	$\binom{n}{i,j,k}$	trinomial coefficient	p.89
$\mathbb{Z}$	the integers	p.ix	sinh	hyperbolic sine	p.364
$\mathbb{N}$	the natural numbers	p.ix	cosh	hyperbolic cosine	p.364
$\mathbb{C}$	the complex numbers	p.9	tanh	hyperbolic tangent	p.364
$n!$	factorial	p.xi	$[a,b]$	interval $a \leqslant x \leqslant b$	p.341
$\lfloor x \rfloor$	integer part (or floor)	p.78	(a,b)	interval $a < x < b$	p.341

Relating to Complex Numbers

Notation	Meaning	Page	Notation	Meaning	Page
i	$\sqrt{-1}$	p.9	$\lvert z \rvert$	modulus of z	p.14
$\operatorname{Re} z$	real part of z	p.9	$\arg z$	argument of z	p.15
$\operatorname{Im} z$	imaginary part of z	p.9	$\operatorname{cis}\theta$	$\cos\theta + i\sin\theta$	p.15
$\bar{z}$	conjugate of z	p.11	$\sqrt{z}$	square roots of z	p.23

Relating to Vectors and Matrices

Notation	Meaning	Page	Notation	Meaning	Page
$\mathbb{R}^n$	space of $1 \times n$ row vectors	p.129	$\mathbf{e}_i$	standard basis vector in $\mathbb{R}^n$	p.131
$\mathbb{R}_n$	space of $n \times 1$ column vectors	p.129	$\mathbf{i}, \mathbf{j}, \mathbf{k}$	standard basis in $\mathbb{R}^3$	p.131
M_{mn}	set of $m \times n$ real matrices	p.147	E_{ij}	standard basis vector in M_{mn}	p.154

Notation	Meaning	Page	Notation	Meaning	Page
Row(A)	rowspace of A	p.181	$A=(a_{ij})$	the (i,j)th entry of A is a_{ij}	p.146
Col(A)	column space of A	p.204	$[A]_{ij}$	(i,j)th entry of A	p.146
Null(A)	the null space of A	p.204	A^T	the transpose of A	p.153
dim	dimension	p.199	A^{-1}	the inverse of A	p.165
rank(A)	rank of A	p.184	$\mathbf{x}\cdot\mathbf{y}$	scalar or dot product	p.135
RRE(A)	reduced row echelon form of A	p.182	$\mathbf{x}\wedge\mathbf{y}$	vector or cross product	p.248
I_n	$n\times n$ identity matrix	p.150	$\oplus$	direct sum	p.270
0_{mn}	$m\times n$ zero matrix	p.147	$\det A$	determinant of A	p.223
δ_{mn}	Kronecker delta	p.150	R_θ	anticlockwise rotation by θ	p.212
S_{ij}	ERO swapping ith and jth rows	p.172	S_θ	reflection in $y=x\tan\theta$	p.212
$M_i(\lambda)$	ERO multiplying ith row by λ	p.172	${}_{\mathcal{W}}T_{\mathcal{V}}$	matrix of T wrt bases $\mathcal{V}$ and $\mathcal{W}$	p.284
$A_{ij}(\lambda)$	ERO adding λ(row i) to row j	p.172	$J(\lambda,r)$	Jordan block	p.160
$\langle S\rangle$	the span of S	p.196	$Z(\mathbf{v},A)$	cyclic subspace	p.335
$(\mathbf{c}_1\vert\cdots\vert\mathbf{c}_n)$	matrix with columns $\mathbf{c}_1,\ldots,\mathbf{c}_n$	p.196	$C(f)$	companion matrix	p.335
$(\mathbf{r}_1/\cdots/\mathbf{r}_m)$	matrix with rows $\mathbf{r}_1,\ldots,\mathbf{r}_m$	p.196	K_n	complete graph	p.275
diag($a,\ldots,z$)	diagonal matrix	p.151	$K_{m,n}$	complete bipartite graph	p.275
diag(A,B)	matrix with blocks A,B	p.151			

Relating to Special Functions

Notation	Meaning	Page	Notation	Meaning	Page
B_n	Bernoulli numbers	p.123	$\operatorname{erf} x$	error function	p.368
C_n	Catalan numbers	p.118	$\operatorname{li}(x)$	logarithmic integral	p.393
F_n	Fibonacci numbers	p.100	$B_n(x)$	Bernoulli polynomials	p.123
H_n	harmonic numbers	p.424	$H_n(x)$	Hermite polynomials	p.123
L_n	Lucas numbers	p.104	$P_n(x)$	Legendre polynomials	p.124
$\Gamma(a)$	gamma function	p.380	$T_n(x), U_n(x)$	Chebyshev polynomials	p.125
$B(a,b)$	beta function	p.377	$L_n(x)$	Laguerre polynomials	p.125
$\psi(a)$	digamma function	p.424	γ	Euler's constant	p.369
$E(k), K(k)$	elliptic integrals	p.419	G	Catalan's constant	p.381

Relating to Integration and Differential Equations

Notation	Meaning	Page	Notation	Meaning	Page
$\mathbf{1}_I(x)$	indicator function	p.342	$\bar{f}(s)$	Laplace transform of $f(x)$	p.467
$[a,b]$	$\{x : a \leqslant x \leqslant b\}$	p.342	$\mathcal{L}$	Laplace transform	p.467
(a,b)	$\{x : a < x < b\}$	p.342	$H(x)$	Heaviside function	p.468
$[a,b)$	$\{x : a \leqslant x < b\}$	p.342	$\delta(x)$	Dirac delta function	p.477
$(a,b]$	$\{x : a < x \leqslant b\}$	p.342			

Abbreviations

Notation	Meaning	Page	Notation	Meaning	Page
LHS	left-hand side		RHS	right-hand side	
RRE	row-reduced echelon form	p.177	DE	differential equation	p.426
ERO	elementary row operation	p.172	ODE	ordinary differential equation	p.426
ECO	elementary column operation	p.192	PDE	partial differential equation	p.432

Notation	Meaning	Page	Notation	Meaning	Page
FTC	fundamental theorem of calculus	p.354	SHM	simple harmonic motion	p.450
IBP	integration by parts	p.372	pdf	probability density function	p.413
			cdf	cumulative distribution function	p.413

The Greek Alphabet

A, α	alpha	H, η	eta	N, ν	nu	T, τ	tau
B, β	beta	Θ, θ	theta	Ξ, ξ	xi	Y, υ	upsilon
Γ, γ	gamma	I, ι	iota	O, o	omicron	Φ, ϕ	phi
Δ, δ	delta	K, κ	kappa	Π, π	pi	X, χ	chi
E, ϵ	epsilon	Λ, λ	lambda	P, ρ	rho	Ψ, ψ	psi
Z, ζ	zeta	M, μ	mu	$\Sigma, \sigma, \varsigma$	sigma	Ω, ω	omega