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Introduction

1.1 Dimensional Homogeneity

Most physical variables and constants have dimensions. A mass, a distance, and a time have, respectively, the dimensions *mass*, *length*, and *time*. Dimensions are often taken for granted. They slip beneath our notice. Yet dimensions are important elements of the way we think about the physical world.

Imagine an animal that grows in size while keeping roughly the same shape. Galileo (1584–1642) reasoned that the weight of the animal increases with its volume in direct proportion to the third power l^3 of a characteristic length l, say, for an elephant, the length of its foreleg. Because animal limbs push and pull across a cross-section, the strength of an animal increases with the cross-sectional area of its limbs, that is, in direct proportion to l^2. Thus the animal's strength-to-weight ratio changes in proportion as l^2/l^3, that is, as $1/l$ or l^{-1}. Therefore, larger animals are less able to support their weight than smaller ones are. Galileo illustrated this conclusion by comparing the relative strength of dogs and horses – creatures with roughly the same shape.

> A small dog could probably carry on his back two or three dogs of his own size; but I believe that a horse could not carry even one of his own size. [1]

If Galileo had not thought dimensionally, he could not have made this interesting argument.

By the time of Isaac Newton (1643–1727) scientists had begun to think in terms of combinations of different dimensions. For instance, the dimension of speed is *length* divided by *time*, the dimension of acceleration is *length* divided by *time squared*, and, according to Newton's second law, the dimension of force is *mass* times *length* divided by *time squared*. Newton regarded *mass, length*, and *time* as primary, fundamental dimensions and combinations of these as secondary, derived ones. [2]

One of the first things a physics student learns is that one should not add, subtract, equate, or compare quantities with different dimensions or quantities with the same dimension and different units of measure. For instance, one cannot add a mass to a length or, for that matter, 5 meters to 2 kilometers. This rule against what is sometimes called “adding apples and oranges” means that every term that is added, subtracted, equated, or compared in every valid equation or inequality must be of the same dimension denominated in the same unit of measure. This is the *principle of dimensional homogeneity*.

The principle of dimensional homogeneity is nothing new. Scientists have long assumed that every term in every fully articulated equation that accurately describes a physical state or process has the same dimension denominated in the same unit of measure. However, it was not until 1822 that Joseph Fourier (1768–1830) expressed this principle in a way that allowed important consequences to be derived from it. [3]

Symmetry under Change of Units

Behind the principle of dimensional homogeneity is a symmetry principle. Symmetry principles tell us that something remains the same as something else is changed. Here the *something that remains the same* is the form of the equation or inequality and the *things that are changed* are the units in which the dimensions of its terms are expressed. Thus, if we change the unit of length from meters to kilometers and the form of the equation does not change, that equation observes this particular symmetry and is, at least in this regard, dimensionally homogeneous. If we change all of its units and the form of the equation does not change, this equation is fully dimensionally homogeneous.

An equation can be useful without being dimensionally homogeneous. For instance,

$$s = 4.9t^2 \tag{1.1}$$

correctly describes the downward displacement s denominated in meters of an object falling freely from rest for a period of time t denominated in seconds. Yet (1.1) is not invariant with respect to changes in its units of measure. Compare (1.1) with

$$s = \frac{1}{2}gt^2 \tag{1.2}$$

in which we have parameterized the acceleration of gravity with the symbol g. This equation is now symmetric with respect to changes in all its units of measure. It is fully articulated and dimensionally homogeneous.

We are concerned in this text with relations among dimensional variables, for instance, s and t, and dimensional constants, such as g, that are symmetric with respect to changes in units and, therefore, dimensionally homogeneous. The principle of dimensional homogeneity and its consequences are foundational to the theory of dimensional analysis.

1.2 Dimensionless Products

Consider the vertical position y of a freely falling object at time t. We know that

$$y - y_o = \upsilon_{yo} t - \frac{gt^2}{2} \tag{1.3}$$

where $y - y_o$ is the object's displacement from its initial position y_o, υ_{yo} is its initial velocity, g is the magnitude of the gravitational acceleration, and our coordinate system is oriented so that y becomes more negative as the object falls and time advances. Equation (1.3) observes the principle of dimensional homogeneity. For the dimension of $y - y_o$ is *length*; the dimensions of υ_{yo} and t are, respectively, *length/time* and *time* so that the dimension of their product $\upsilon_{yo} t$ is *length*; and the dimension of gt^2 is *length/time*^{2} multiplied by *time*^{2} or, again, *length*. Furthermore, (1.3) contains no dimensional constants masquerading as dimensionless numbers – as does (1.1).

One consequence of the dimensional homogeneity of (1.3) is that dividing each of its terms by gt^2 produces an equation,

$$\frac{y - y_o}{gt^2} = \frac{\upsilon_{yo}}{gt} - \frac{1}{2}, \tag{1.4}$$

that relates one dimensionless combination or "product" $(y - y_o)/gt^2$ to another υ_{yo}/gt. This transformation of a dimensionally homogeneous equation, from a relation among dimensional variables and constants to a relation among dimensionless products, can always be realized.

Consider, for instance, the Stefan-Boltzmann law, according to which the density of radiant energy E in a cavity of volume V whose walls are at a temperature T is described by

$$\frac{E}{V} = \frac{8\pi^5}{15} \frac{k_B^4 T^4}{c^3 h^3} \tag{1.5}$$

where k_B is Boltzmann's constant, c is the speed of light, and h is Planck's constant. Each of the variables E, V, and T and each of the constants k_B, c, and

h are dimensional quantities. If equation (1.5) is dimensionally homogeneous (and it is), it may take the form

$$\frac{Ec^3h^3}{Vk_B^4T^4} = C \tag{1.6}$$

of a dimensionless product $Ec^3h^3/\left(Vk_B^4T^4\right)$ equal to a dimensionless number C. In this case $C = 8\pi^5/15$.

Dimensional Analysis

We have, in these two examples, turned dimensionally homogeneous relations among dimensional variables and constants, (1.3) and (1.5), into relations among one or more dimensionless products, (1.4) and (1.6). We shall soon learn a way to reverse this process. We will first use an algorithm, the Rayleigh algorithm, to discover the dimensionless products relevant to a particular state or process. When only one dimensionless product is found, the only way it can form a dimensionally homogeneous equation is for this product to equal some dimensionless number as in (1.6). When two or more dimensionless products are found, as in (1.3), they must be related to one another by some function, as in (1.4). The Rayleigh algorithm does not determine these numbers and these functions but merely finds the dimensionless products.

1.3 Dimensional Formulae

Every dimensional variable or constant assumes values in the form of a number times a unit of measure – for instance, 5 kilograms or 16 meters. Furthermore, every unit of measure makes its dimension known. A meter per second and a kilometer per hour are both a *length/time* while a metric ton and a kilogram are both *masses*. We need to know the dimension, more precisely the dimensional formula, of every relevant dimensional variable and constant in order to dimensionally analyze a state or process. For this purpose we use the symbol M to stand for the dimension *mass*, L to stand for the dimension *length*, and T to stand for the dimension *time*. The notation $[x]$ means "the dimension of x." Therefore, $[m] = M$ and $[g] = LT^{-2}$ are dimensional formulae. Not every dimensional formula can be expressed in terms of only M, L, and T, but many can be.

The dimensional formula of a product of factors is the product of the dimensional formula of each factor. Thus $[ma] = [m][a]$. For convenience, we define the dimensional formula of a dimensionless number to be 1. Therefore, $[\pi] = 1$ and so $[9.8 \cdot m/s^2] = [9.8] \cdot [m/s^2] = [m] \cdot [s^{-2}] = LT^{-2}$.

1.4 The Rayleigh Algorithm

John William Strutt (1842–1919), also known as Lord Rayleigh, successfully applied dimensional analysis to a number of problems over a long career. He dimensionally analyzed the strength of bridges, the velocity of waves on the surface of water, the vibration of tuning forks and drops of falling water, the color of the sky, the decay of charge on an electrical circuit, the determinants of viscosity, and the flow of heat from a hot object immersed in a cool stream of water. Rayleigh prefaced his 1915 summary of these applications of the principle of dimensional homogeneity (known to him as "the principle of similitude") with these words,

> I have often been impressed with the scanty attention paid even by original workers to the great principle of similitude. It happens not infrequently that results in the form of "laws" are put forth as novelties on the basis of elaborate experiments, which might have been predicted *a priori* after a few minutes' consideration. [4]

While our applications of dimensional analysis may require more than "a few minutes consideration," Rayleigh's method of applying "the principle of similitude" is simple and direct. We adopt it, as have many others, with only slight modification.

A Marble on the Interior Surface of a Cone

To illustrate Rayleigh's method, imagine a small marble of mass m rolling in a circle of radius R on the interior surface of an inverted cone defined by an angle θ as illustrated in Figure 1.1. We wish to know how the time Δt required for the marble to complete one orbit is determined by m, R, and θ. The acceleration of gravity g may also enter into the relation we seek. Gravity,

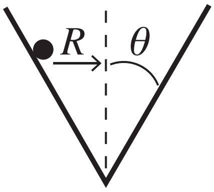

Figure 1.1. Marble on the interior surface of a cone defined by angle θ.

after all, is one of the two forces that keep the marble on the cone's surface. The intermolecular forces of the material composing the cone and the marble also determine, in some degree, the period Δt, but we ignore these forces because we believe their effect is adequately accounted for by assuming the marble stays on the surface of the cone. By including some variables and constants in our analysis and excluding others we construct a model of the marble's motion.

Rayleigh's Algorithm

Rayleigh's method of dimensional analysis identifies the dimensionless products one can form out of the model variables and constants, in this case Δt, m, R, g, and θ. Each dimensionless product takes a form $\Delta t^{\alpha} m^{\beta} R^{\gamma} g^{\delta} \theta^{\varepsilon}$ determined by the Greek letter exponents α, β, γ, δ, and ε or, somewhat more simply, by the form $\Delta t^{\alpha} m^{\beta} R^{\gamma} g^{\delta}$ and the exponents α, β, γ, and δ. After all, the angle θ, whether denominated in radians or degrees, is proportional to a ratio of an arc length to a radius, that is, a ratio of one length to another. While angles have units (degrees or radians), their units are dimensionless.

The key to Rayleigh's method of finding dimensionless products is to require that the product $\Delta t^{\alpha} m^{\beta} R^{\gamma} g^{\delta}$ be dimensionless. Since

$$\begin{aligned}\left[\Delta t^{\alpha} m^{\beta} R^{\gamma} g^{\delta}\right] &= [\Delta t^{\alpha}]\left[m^{\beta}\right][R^{\gamma}]\left[g^{\delta}\right] \\ &= [\Delta t]^{\alpha}[m]^{\beta}[R]^{\gamma}[g]^{\delta} \\ &= T^{\alpha} M^{\beta} L^{\gamma}\left(LT^{-2}\right)^{\delta} \\ &= T^{\alpha-2\delta} M^{\beta} L^{\gamma+\delta},\end{aligned} \tag{1.7}$$

the product $\Delta t^{\alpha} m^{\beta} R^{\gamma} g^{\delta}$ is dimensionless when

$$T : \alpha - 2\delta = 0, \tag{1.8a}$$

$$M : \beta = 0, \tag{1.8b}$$

and

$$L : \gamma + \delta = 0. \tag{1.8c}$$

The three equations (1.8a), (1.8b), and (1.8c) constrain the four unknowns, α, β, γ, and δ, to a family of solutions $\beta = 0$, $\gamma = -\alpha/2$, and $\delta = \alpha/2$ parameterized by α. [The symbols T, M, and L preceding equations (1.8) identify the source of each constraint.] Therefore, $\left(\Delta t g^{1/2}/R^{1/2}\right)^{\alpha}$ is dimensionless for any α, which means that $\Delta t g^{1/2}/R^{1/2}$, as well as θ, is dimensionless.

Once we know the dimensionless products that can be formed out of the model's dimensional variables and constants, we know they must be related to one another by an undetermined function, that is, in this case expressed by

$$\Delta t = \sqrt{\frac{R}{g}} \cdot f(\theta) \tag{1.9}$$

where $f(\theta)$ is a dimensionless function of the dimensionless "product" θ. This is as far as dimensional analysis *per se* takes us. A more detailed, dynamical study reveals that $f(\theta) = 2\pi\sqrt{\tan\theta}$.

The Rayleigh Algorithm Modified

Note that equations (1.8) are solved by a family of solutions parameterized by a non-vanishing exponent α and also that the identity of the dimensionless product $\Delta t g^{1/2}/R^{1/2}$ is independent of the value of α. The exponent α, introduced in the term Δt^{α}, seems superfluous, and indeed, it is – as long as we know that Δt, the variable whose expression we seek, the *variable of interest*, remains in the dimensionless product. In this case no harm is done by freely choosing α. In particular, choosing $\alpha = 1$ is equivalent to determining the three remaining exponents β, γ, and δ as those that make $\Delta t m^{\beta} R^{\gamma} g^{\delta}$ dimensionless. Then $\beta = 0$, $\gamma = -1/2$, and $\delta = 1/2$. This solution again produces the dimensionless product $\Delta t g^{1/2}/R^{1/2}$. Henceforth we adopt the practice of including the variable of interest with an exponent of 1 as the first factor in the dimensionless product.

Observe that this analysis, issuing as it does in $\Delta t g^{1/2}/R^{1/2} = f(\theta)$, is a significant advance on knowing only that Δt, m, R, g, and θ are related to one another by an unknown function, say, by $\Delta t = h(m, R, g, \theta)$. For suppose that empirically determining the function $f(\theta)$ in (1.9) requires 10 pairs of $\Delta t g^{1/2}/R^{1/2}$ versus θ data. Since 10 pairs of data determine how one term depends on one other (the others remaining constant), 10^4 pairs of data are required to determine how one variable Δt depends on the four dimensional variables and constants m, R, g, and θ. Thus, 10^4 pairs of data are required to determine the function in $\Delta t = h(m, R, g, \theta)$. The Rayleigh algorithm reduces the effort required by a factor of 1,000!

1.5 The Buckingham π Theorem

In 1914, Edgar Buckingham (1867–1940) proved, in formal algebraic detail, a theorem we have, thus far, merely illustrated – a theorem usually referred to as the *Buckingham π theorem* or sometimes, more simply, as the *π theorem*. [5] The π theorem may be divided into two conceptually distinct parts. First,

> If an equation is dimensionally homogeneous, it can be reduced to a relationship among a complete set of independent dimensionless products. [6]

A set of dimensionless products is *complete* if and only if all possible dimensionless products of the dimensional variables and constants can be expressed as a product of powers of the members of this set. The members of this set are *independent* if and only if none of them can be expressed as a product of powers of the other members.[a] The symbol π in the phrase π *theorem* refers to members of a complete set of independent dimensionless products. Buckingham denoted these dimensionless products by $\pi_1, \pi_2 \ldots$ Thus, for example, in the marble on the interior of a cone problem, $\pi_1 = \Delta t g^{1/2}/R^{1/2}$ and $\pi_2 = \theta$. The second part of the π theorem consists of the following statement.

> The number of complete and independent dimensionless products N_p is equal to the number of dimensional variables and constants N_V that describe the state or process minus the minimum number of dimensions N_D needed to express their dimensional formulae. Thus,

$$N_p = N_V - N_D. \tag{1.10}$$

Statement (1.10) is the most common expression of the π theorem.

1.6 The Number of Dimensions

Most dimensional analysts adopt M, L, and T as dimensions appropriate for mechanical processes and states. We did so in describing the marble on the interior surface of a cone. In that case, $N_D = 3$. Furthermore, since Δt, m, R, g, and θ describe the marble's motion, $N_V = 5$. Therefore, according to (1.10) $N_P = N_V - N_D$, $2(= 5 - 3)$ complete and independent dimensionless products should be produced. By applying the Rayleigh algorithm we find these to be $\Delta t g^{1/2}/R^{1/2}$ and θ. The set $\Delta t g^{1/2}/R^{1/2}$ and θ is complete because every possible dimensionless product of Δt, m, R, g, and θ can be expressed as a product of some power of $\Delta t^2 R/g$ times some power of θ. And its members are independent because $\Delta t^2 R/g$ and θ are not powers of each other.

But the *minimum* number of dimensions needed to express the dimensional formulae of the N_V dimensional variables and constants is not always 3, as it is in this example. Neither is the identity of the minimum number of dimensions necessarily M, L, and T – as they often are in mechanical problems. Rather, the dimensions required are, in Buckingham's words, the "arbitrary fundamental

[a] A complete set of independent products plus all dimensionless products that can be formed from them is itself a *group* because these products: (a) are closed under multiplication, (b) contain an identity element 1, and (c) each product π_i has an inverse π_i^{-1}.

units [dimensions] needed as a basis for the absolute system." [7] And only when we can ensure that N_D is the *minimum* number of dimensions needed can we depend on $N_P = N_V - N_D$ to be observed. Otherwise, $N_P = N_V - N_D$ remains a mere "rule of thumb" – often observed but sometimes not. We will learn how to recognize the minimum number of dimensions in Section 2.2.

1.7 The Number of Dimensionless Products

Note that the more dimensionless products, π_1, π_2, ... π_{N_P}, produced, the less determined the state or process described by the dimensional model. After all, if only one product π_1 is produced, the result sought assumes a form $f(\pi_1) = 0$ whose solution $\pi_1 = C$ is in terms of a single undetermined dimensionless number C. However, if two dimensionless products, π_1 and π_2, are produced, these are related by $g(\pi_1, \pi_2) = 0$ whose solution $\pi_1 = h(\pi_2)$ leaves a function $h(\pi_2)$ of a single variable undetermined. And if three dimensionless products, π_1, π_2, and π_3, are produced, these are related by a function $j(\pi_1, \pi_2, \pi_3) = 0$ whose solution $\pi_1 = k(\pi_2, \pi_3)$ leaves a function $k(\pi_1, \pi_2)$ of two variables undetermined.

It is clear that in order to more completely determine a state or process, we need to minimize the number N_P of complete and independent dimensionless products. According to the rule of thumb $N_P = N_V - N_D$, we do this by minimizing N_V (the number of dimensional variables and constants that describe the model) and, assuming we have such freedom, by maximizing N_D (the minimum number of dimensions in terms of which these variables and constants can be expressed.) However, minimizing N_V and maximizing N_D are not straightforward tasks. Both require skill and judgment – the same kind of skill and judgment needed to construct a model of a physical state or process.

1.8 Example: Pressure of an Ideal Gas

Many of these ideas are illustrated in the dimensional analysis of how the pressure p of an ideal gas depends on quantities that describe its state. The pressure a gas exerts on its container walls is the average rate per unit area with which its molecules collide with and transfer momentum to the wall. The ideal gas model treats these molecules as randomly and freely moving, massive, point particles whose instantaneous collisions with other particles and with the walls conserve their energy.

Table 1.1

Symbol	Description	Dimensional Formula
p	Pressure	$ML^{-1}T^{-2}$
N/V	Number density	L^{-3}
m	Molecular mass	M
$\overline{v}$	Characteristic speed	LT^{-1}

Therefore, we believe the ideal gas pressure p should depend on the number density of the gas molecules N/V where N is the number of gas molecules contained in volume V, the mass of each of the molecules m, and their average or characteristic speed $\overline{v}$. These parameters should be sufficient, since they are the elements out of which the momentum of the gas particles, the rate at which they collide with the wall, and their energy are composed. To include other variables or constants such as, for instance, the acceleration of gravity g would be to introduce extraneous dimensionless products and make our result not so much inaccurate as uninformative. For convenient reference, we collect these symbols, their descriptions, and their dimensional formulae in Table 1.1.

Note that we have included the number of particles N in volume V only in the combination N/V. For this reason, we have $4(=N_V)$ variables: p, N/V, m, and $\overline{v}$. Since they are expressed in terms of $3(=N_D)$ dimensions, M, L, and T, the rule of thumb $N_P = N_V - N_d$ predicts 1 $(= 4 - 3)$ dimensionless product.

Recall that in executing the Rayleigh algorithm, here and elsewhere, we enter the variable of interest, the one whose expression in terms of other variables we seek, in this case the gas pressure p, with an exponent of 1 in the first position of the product $p(N/V)^{\alpha}m^{\beta}\overline{v}^{\gamma}$. Then we find the three exponents, α, β, and γ, that render this product dimensionless. Thus,

$$\begin{aligned}\left[p(N/V)^{\alpha}m^{\beta}\overline{v}^{\gamma}\right] &= \left(ML^{-1}T^{-2}\right)\left(L^{-3}\right)^{\alpha}M^{\beta}\left(LT^{-1}\right)^{\gamma} \\ &= M^{1+\beta}L^{-1-3\alpha+\gamma}T^{-2-\gamma}\end{aligned} \tag{1.11}$$

and, therefore, the exponents must be solutions of

$$M : 1 + \beta = 0, \tag{1.12a}$$

$$L : -1 - 3\alpha + \gamma = 0, \tag{1.12b}$$

and

$$T : -2 - \gamma = 0. \tag{1.12c}$$