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A Qualitative Overview of MHD

> The neglected borderland between two branches of knowledge is often that which best repays cultivation, or, to use a metaphor of Maxwell's, the greatest benefits may be derived from a cross-fertilisation of the sciences.
>
> *Rayleigh, 1884*

1.1 What Is MHD?

Magnetic fields influence many natural and man-made flows. They are routinely used in industry to heat, pump, stir and levitate liquid metals. There is the terrestrial magnetic field which is maintained by fluid motion in the earth's molten core, the solar magnetic field, which generates sunspots and solar flares, and the interplanetary magnetic field which spirals outward from the sun, carried by the solar wind. The study of these flows is called magnetohydrodynamics (MHD, for short). Formally, MHD is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic[1], which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes.

The mutual interaction of a magnetic field, **B**, and a velocity field, **u**, arises partially as a result of the laws of Faraday and Ampère, and partially because of the Lorentz force experienced by a current-carrying body. The exact form of this interaction is analysed in detail in the following chapters, but perhaps it is worth stating now, without any form of proof, the nature of this coupling. It is convenient, although somewhat artificial, to split the process into three parts.

(i) The relative movement of a conducting fluid and a magnetic field causes an electromotive force (EMF) (of order $|\mathbf{u} \times \mathbf{B}|$) to develop in accordance with Faraday's law of induction. In general, electrical currents will ensue, the current density being of order $\sigma(\mathbf{u} \times \mathbf{B})$, σ being the electrical conductivity.

[1] The study of magnetically polarised fluids is called ferrohydrodynamics, and such fluids are referred to as magnetic fluids.

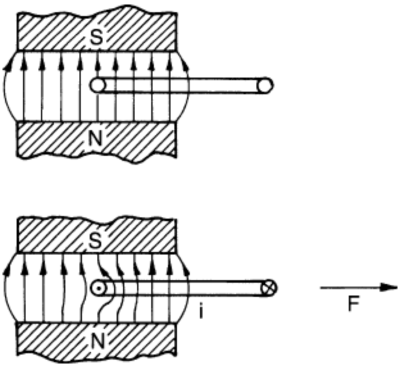

Figure 1.1 Interaction of a magnetic field and a moving wire loop.

(ii) These induced currents must, according to Ampère's law, give rise to a second, induced magnetic field. This adds to the original magnetic field and the change is usually such that the fluid appears to 'drag' the magnetic field lines along with it.

(iii) The combined magnetic field (imposed plus induced) interacts with the induced current density, **J**, to give rise to a Lorentz force (per unit volume) of **J** × **B**. This acts on the conductor and is generally directed so as to inhibit the relative movement of the magnetic field and the fluid.

Note that these last two effects have similar consequences. In both cases the relative movement of fluid and field tends to be reduced. Fluids can 'drag' magnetic field lines (effect (ii)) and magnetic fields can pull on conducting fluids (effect (iii)). It is this partial 'freezing together' of the medium and the magnetic field which is the hallmark of MHD.

These effects are, perhaps, more familiar in the context of conventional electrodynamics. Consider a wire loop which is pulled through a magnetic field as shown in Figure 1.1. As the wire loop is pulled to the right, an EMF of order $|\mathbf{u} \times \mathbf{B}|$ is generated which drives a current as shown (effect (i)). The magnetic field associated with the induced current perturbs the original magnetic field and the net result is that the magnetic field lines seem to be dragged along by the wire (effect (ii)). The current also gives rise to a Lorentz force, **J** × **B**, which acts on the wire in a direction opposite to that of the motion (effect (iii)). Thus it is necessary to provide a force, **F**, to move the wire. In short, the wire appears to drag the field lines while the magnetic field reacts back on the wire, tending to oppose the relative movement of the two.

Let us consider effect (ii) in a little more detail. As we shall see later, the extent to which a velocity field influences an imposed magnetic field depends on the product of (i) the typical velocity of the motion, (ii) the conductivity of the fluid, and (iii) the characteristic length scale, ℓ, of the motion. Clearly, if the fluid is non-conducting or the velocity negligible there will be no significant induced magnetic field. (Consider the wire shown in Figure 1.1. If it is a poor conductor or moves very slowly, then the induced current and the associated magnetic field will be weak.) Conversely if σ or **u** is (in some sense) large, then the induced magnetic field may substantially alter the imposed field. The reason ℓ is important is a little less obvious but may be clarified by the following argument. The EMF generated by a relative movement of the imposed magnetic field and the conducting medium is of order $|\mathbf{u} \times \mathbf{B}|$ and so, by Ohm's law, the induced current density is of the order of $\sigma|\mathbf{u} \times \mathbf{B}|$. However, a modest current density spread over a large area can produce a strong magnetic field, whereas the same current density spread over a small area induces only a weak magnetic field. It is therefore the product $\sigma \mathbf{u} \ell$ which determines the ratio of the induced field to the applied magnetic field. In the limit $\sigma \mathbf{u} \ell \to \infty$ (typical of so-called ideal conductors), the induced and imposed magnetic fields are of the same order. In such cases it turns out that the combined magnetic field behaves as if it were locked into the fluid. Conversely, when $\sigma \mathbf{u} \ell \to 0$, the imposed magnetic field remains relatively unperturbed. Astrophysical MHD tends to be closer to the first situation, not so much because of the high conductivity of the plasmas involved, but because of the vast characteristic length scale. Liquid-metal MHD, on the other hand, usually lies closer to the second limit, with **u** leaving **B** relatively unperturbed. Nevertheless, it should be emphasised that effect (iii) is still strong in liquid metals, so that an imposed magnetic field can substantially alter the velocity field.

Perhaps it is worth taking a moment to consider the case of liquid metals in a little more detail. They have a reasonable conductivity ($\sim 10^6 \Omega^{-1} \text{m}^{-1}$) but the velocity involved in a typical laboratory or industrial process is low (~1 m/s). As a consequence, the induced current densities are generally rather modest (a few Amps per cm^2). When this is combined with a small length scale (~0.1 m in the laboratory) the induced magnetic field is usually found to be negligible by comparison with the imposed field. There is very little 'freezing together' of the fluid and the magnetic field. However, the imposed magnetic field is often strong enough for the Lorentz force, $\mathbf{J} \times \mathbf{B}$, to dominate the motion of the fluid. We tend to think of the coupling as being one-way: **B** controls **u** through the Lorentz force, but **u** does not substantially alter the imposed field, **B**. There are, however, exceptions. Perhaps the most important of these is the earth's dynamo. Here, motion in the liquid-metal core of the earth twists, stretches and intensifies the terrestrial magnetic field, maintaining it against the natural processes of decay. It is the large length scales which are important here. While the induced current densities are weak, they are spread over a large area and as a result their combined effect is to induce a substantial magnetic field.

In summary, then, the freezing together of the magnetic field and the medium is usually strong in astrophysics, significant in geophysics, weak in metallurgical

MHD, and utterly negligible in electrolytes. However, the influence of **B** on **u** can be important in all four situations.

1.2 A Brief History of MHD

The laws of magnetism and fluid flow are hardly a twentieth-century innovation, yet MHD became a fully fledged subject only in the late 1930s or early 1940s. The reason, probably, is that there was little incentive for nineteenth-century engineers to capitalise on the possibilities offered by MHD. Thus, while there were a few isolated experiments by nineteenth-century physicists such as Faraday (he tried to measure the voltage across the Thames induced by its motion through the earth's magnetic field), the subject languished until the turn of the century. Things started to change, however, when astrophysicists realised just how ubiquitous magnetic fields and plasmas are throughout the universe. This culminated in 1942 with the discovery of the Alfvén wave, a phenomenon which is peculiar to MHD and important in astrophysics. (A magnetic field line can transmit transverse inertial waves, just like a plucked string.) Around the same time, geophysicists began to suspect that the earth's magnetic field was generated by dynamo action within its liquid-metal core, an hypothesis first put forward in 1919 by Larmor in the context of the sun's magnetic field. A period of intense research followed and continues to this day.

Plasma physicists, on the other hand, acquired an interest in MHD in the 1950s as the quest for controlled thermonuclear fusion gathered pace. They were particularly interested in the stability, or lack of stability, of plasmas confined by magnetic fields, and great advances in stability theory were made as a result. Indeed, stability techniques developed in the 1950s and 1960s by the plasma physicists have since found application in many other branches of fluid mechanics.

The development of MHD in engineering was slower and did not really get going until the 1960s. However, there was some early pioneering work by the engineer J. Hartmann, who invented the electromagnetic pump in 1918. Hartmann also undertook a systematic theoretical and experimental investigation of the flow of mercury in a homogeneous magnetic field. In the introduction to the 1937 paper describing his researches he observed:

> The invention [his pump] is, as will be seen, no very ingenious one, the principle utilised being borrowed directly from a well-known apparatus for measuring strong magnetic fields. Neither does the device represent a particularly effective pump, the efficiency being extremely low due mainly to the large resistivity of mercury and still more to the contact resistance between the electrodes and the mercury. In spite hereof considerable interest was in the course of time bestowed on the apparatus, firstly because of a good many practical applications in cases where the efficiency is of small moment and then, during later years, owing to its inspiring nature. As a matter of fact, the study of the pump revealed to the

author what he considered a new field of investigation, that of flow of a conducting liquid in a magnetic field, a field for which the name Hg-dynamics was suggested.

The name, of course, did not stick, but we may regard Hartmann as the father of liquid-metal MHD, and, indeed, the term ‘Hartmann flow’ is now used to describe duct flows in the presence of a magnetic field. Despite Hartmann’s early researches, it was only in the early 1960s that MHD began to be exploited in engineering. The impetus for change came largely as a result of three technological innovations: (i) fast-breeder reactors use liquid sodium as a coolant, and this needs to be pumped; (ii) controlled thermonuclear fusion requires that the hot plasma be confined away from material surfaces by magnetic forces; and (iii) MHD power generation, in which ionised gas is propelled through a magnetic field, was thought to offer the prospect of improved power station efficiencies. This last innovation turned out to be quite impracticable, and its failure was rather widely publicised in the scientific community. However, as the interest in power generation declined, research into metallurgical MHD took off. Three decades later, magnetic fields are routinely used to heat, pump, stir and levitate liquid metals in the metallurgical industries. The key point is that the Lorentz force provides a non-intrusive means of controlling the flow of metals. With constant commercial pressure to produce cheaper, better and more consistent materials, MHD provides a unique means of exercising greater control over casting and refining processes.

So there now exist at least four overlapping communities who study MHD. Astrophysicists are concerned with the galactic magnetic field, the behaviour of (magnetically active) accretion discs, and the dynamics of stars. Planetary scientists study the generation of magnetic fields within the interior of planets, while plasma physicist are interested in the behaviour of magnetically confined plasmas. Finally, engineers study liquid-metal MHD, mostly in the context of the metallurgical industries. These communities have, of course, many common aims and problems, but they tend to use rather different vocabularies and occasionally have different ways of conceiving the same phenomena.

1.3 From Electrodynamics to MHD: A Simple Experiment

Now, the only difference between MHD and conventional electrodynamics lies in the fluidity of the conductor. This makes the interaction between **u** and **B** more subtle and difficult to quantify. Nevertheless, many of the important features of MHD are latent in electrodynamics and can be exposed by simple laboratory experiments. An elementary grasp of electromagnetism is then all that is required to understand the phenomena. Just such an experiment is described below. First, however, we shall discuss those features of MHD which the experiment is intended to illustrate.

1.3.1 Some Important Parameters in Electrodynamics and MHD

Let us introduce some notation. Let μ be the permeability of free space, σ and ρ denote the electrical conductivity and density of the conducting medium, and ℓ be a characteristic length scale. Three important parameters in MHD are

Magnetic Reynolds number, $R_m = \mu\sigma u\ell$
Alfvén velocity, $v_a = B/\sqrt{\rho\mu}$
Magnetic damping time, $\tau = [\sigma B^2/\rho]^{-1}$

The first of these parameters may be considered as a dimensionless measure of the conductivity, while the second and third quantities have the dimensions of speed and time, respectively, as their names suggest.

Now we have already hinted that magnetic fields behave very differently depending on the conductivity of the medium. In fact, it turns out to be R_m, rather than σ, which is important. Where R_m is large, the magnetic field lines act rather like elastic bands frozen into the conducting medium. This has two consequences. First, the magnetic flux passing through any closed material loop (a loop always composed of the same material particles) tends to be conserved during the motion of the fluid. This is indicated in Figure 1.1. Second, as we shall see, small disturbances of the medium tend to result in near-elastic oscillations, with the magnetic field providing the restoring force for the vibration. In a fluid, this results in Alfvén waves, which turn out to have a frequency of $\varpi \sim v_a/\ell$.

When R_m is small, on the other hand, **u** has little influence on **B**, the induced field being negligible by comparison with the imposed field. The magnetic field then behaves quite differently. We shall see that it is dissipative in nature, rather than elastic, damping mechanical motion by converting kinetic energy into heat via Joule dissipation. The relevant time scale is now the damping time, τ, rather than ℓ/v_a.

All of this is dealt with more fully in Chapters 5, 6 and 7. The purpose of this section is to show how a familiar high school experiment is sufficient to expose these two very different types of behaviour, and to highlight the important roles played by R_m, v_a and τ.

1.3.2 Electromagnetism Remembered

Let us start with a reminder of the elementary laws of electromagnetism. (A more detailed discussion of these laws is given in Chapter 2.) The laws which concern us here are those of Ohm, Faraday and Ampère. We start with Ohm's law.

This is an empirical law which, for stationary conductors, takes the form $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{E}$ is the electric field and $\mathbf{J}$ the current density. We interpret this as $\mathbf{J}$ being proportional to the Coulomb force $\mathbf{f} = q\mathbf{E}$ which acts on the free charge carriers,

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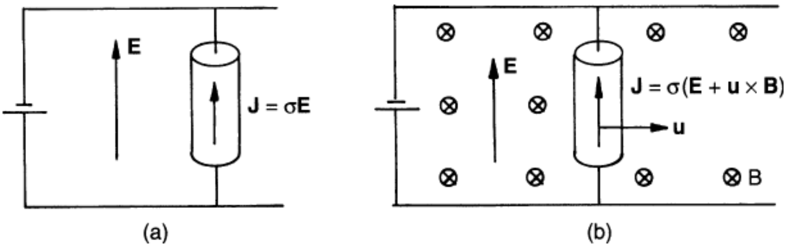

Figure 1.2 Ohm's law in stationary and moving conductors.

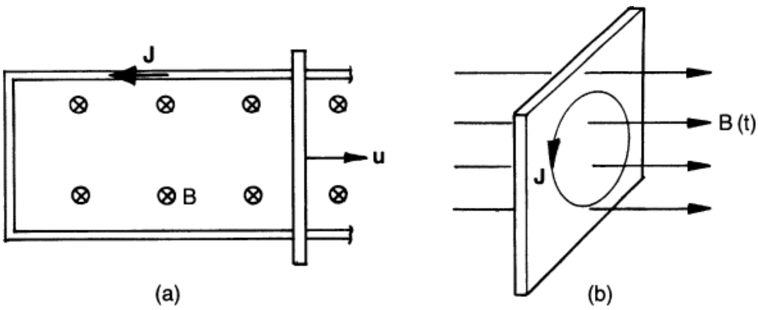

Figure 1.3 Faraday's law: (a) the EMF generated by movement of a conductor, (b) the EMF generated by a time-dependent magnetic field.

q being their charge. If, however, the conductor is moving in a magnetic field with velocity $\mathbf{u}$, the free charges will experience an additional force, $q\mathbf{u} \times \mathbf{B}$, and Ohm's law becomes (Figure 1.2)

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{1.1}$$

The quantity $\mathbf{E} + \mathbf{u} \times \mathbf{B}$, which is the total electromagnetic force per unit charge, arises frequently in electrodynamics and it is convenient to give it a label. We use

$$\mathbf{E}_r = \mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{f}/q. \tag{1.2}$$

Formally, $\mathbf{E}_r$ is the electric field measured in a frame of reference moving with velocity $\mathbf{u}$ relative to the laboratory frame (see Chapter 2). However, for our present purposes it is more useful to think of $\mathbf{E}_r$ as $\mathbf{f}/q$. Some authors refer to $\mathbf{E}_r$ as the *effective electric field.* In terms of $\mathbf{E}_r$, (1.1) becomes $\mathbf{J} = \sigma\mathbf{E}_r$.

Faraday's law tells us about the EMF which is generated in a conductor as a result of (i) a time-dependent magnetic field or (ii) the motion of a conductor within a magnetic field (Figure 1.3). In either case, Faraday's law may be written as

$$\text{EMF} = \oint_C \mathbf{E}_r \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}. \tag{1.3}$$

Here, C is a closed curve composed of line elements $d\mathbf{l}$. The curve may be fixed in space or it may move with the conducting medium (if the medium does indeed move). S is any surface which spans C. (We use the right-hand convention to define the positive directions of $d\mathbf{l}$ and $d\mathbf{S}$.) The subscript on $\mathbf{E}_r$ indicates that we must use the 'effective' electric field for each line element

$$\mathbf{E}_r = \mathbf{E} + \mathbf{u} \times \mathbf{B}, \tag{1.4}$$

where **E, u** and **B** are measured in the laboratory frame and **u** is the velocity of the line element $d\mathbf{l}$.

Next, we need Ampère's law. This (in a round-about way) tells us about the magnetic field associated with a given distribution of current, **J**. If C is a closed curve drawn in space and S is any surface spanning that curve, then Ampère's circuital law states that (Figure 1.4)

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu \int_S \mathbf{J} \cdot d\mathbf{S}. \tag{1.5}$$

Finally, there is the Lorentz force, **F**. This acts on all conductors carrying a current in a magnetic field. It has its origins in the force acting on individual charge carriers, $\mathbf{f} = q(\mathbf{u} \times \mathbf{B})$, and it is easy to show that the force per unit volume of the conductor is given by

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{1.6}$$

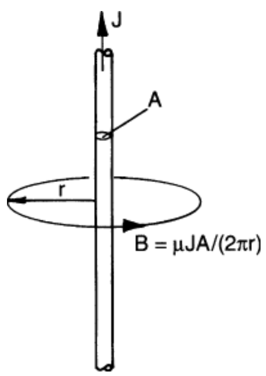

Figure 1.4 Ampère's law applied to a wire.

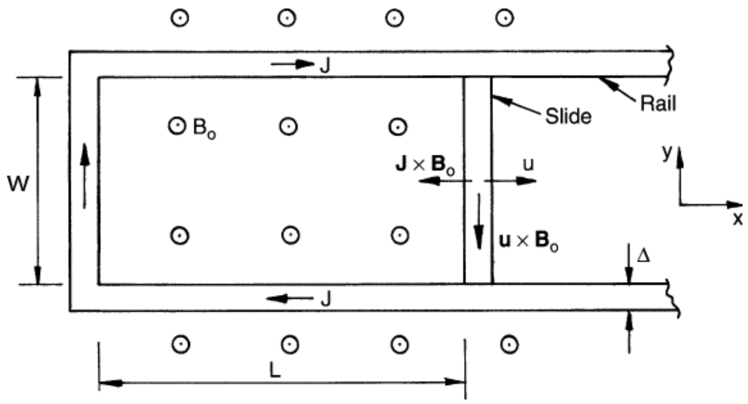

Figure 1.5 A simple experiment for illustrating MHD phenomena.

1.3.3 A Familiar High School Experiment

We now turn to the laboratory experiment. Consider the apparatus illustrated in Figure 1.5. This is frequently used to illustrate Faraday's law of induction. It consists of a horizontal, rectangular circuit sitting in a vertical magnetic field, $\mathbf{B}_0$. The circuit is composed of a frictionless, conducting slide which is free to move horizontally between two rails. We take the rails and slide to have a common thickness Δ and to be made from the same material. To simplify matters, we shall also suppose that the depth of the apparatus is much greater than its lateral dimensions, L and W, so that we may treat the problem as two-dimensional. Also, we take Δ to be much smaller than L or W.

We now show that, if the slide is given a tap, and it has a high conductivity, it simply vibrates as if held in place by a (magnetic) spring. On the other hand, if the conductivity is low, it moves forward as if immersed in treacle, slowing down on a time scale of τ.

Suppose that, at $t = 0$, the slide is given a forward motion, $\mathbf{u}$. This movement of the slide will induce a current density, $\mathbf{J}$, as shown. This, in turn, produces an induced field $\mathbf{B}_i$ which is negligible outside the closed current path but is finite and uniform within the current loop. It may be shown, from Ampère's law, that $\mathbf{B}_i$ is directed downward (Figure 1.6) and has a magnitude and direction given by

$$\mathbf{B}_i = -(\mu \Delta J)\hat{\mathbf{e}}_z. \tag{1.7}$$

Note that the direction of $\mathbf{B}_i$ is such as to try to maintain a constant flux in the current loop (Lenz's law).

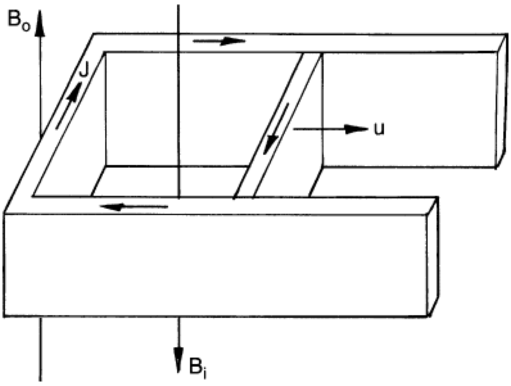

Figure 1.6 Direction of the magnetic field induced by current in the slide.

Next we combine (1.1) and (1.3) to give

$$\frac{1}{\sigma}\oint_C \mathbf{J}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}, \tag{1.8}$$

where C is the material circuit comprising the slide and the return path for $\mathbf{J}$. This yields

$$\frac{d\Phi}{dt} = \frac{d}{dt}\left[LW(B_0 - \mu\Delta J)\right] = 2(L+W)\frac{J}{\sigma}, \tag{1.9}$$

where $\Phi = (B_0 - \mu\Delta J)LW$ is the flux through the circuit. Finally, the Lorentz force (per unit depth) acting on the slide is

$$\mathbf{F} = -J(B_0 - \mu\Delta J/2)\Delta W\hat{\mathbf{e}}_x, \tag{1.10}$$

where the expression in brackets represents the average field within the slide (Figure 1.7). The equation of motion for the slide is therefore

$$\rho\frac{d^2L}{dt^2} = \rho\frac{du}{dt} = -J(B_0 - \mu\Delta J/2), \tag{1.11}$$

where ρ is the density of the metal.

Equations (1.9) and (1.11) are sufficient to determine the two unknown functions $L(t)$ and $J(t)$. Let us introduce some simplifying notation: $B_i = \mu\Delta J$, $\ell = \Delta W/L$, $T = \mu\sigma\Delta W$ and $R_m = \mu\sigma u\ell = uT/L$. Evidently, B_i is the magnitude of the induced field, ℓ is a characteristic length scale, and T is a measure of the conductivity, σ, which happens to have the dimensions of time. Our two equations may be rewritten as