

Contents

<i>Preface</i>	<i>page</i>
1 The Basic Modular Forms of the Nineteenth Century	1
1.1 The Modular Group	1
1.2 Modular Forms	5
1.3 Exercises	11
2 Gauss's Contributions to Modular Forms	13
2.1 Early Work on Elliptic Integrals	13
2.2 Landen and Legendre's Quadratic Transformation	17
2.3 Lagrange's Arithmetic-Geometric Mean	18
2.4 Gauss on the Arithmetic-Geometric Mean	20
2.5 Gauss on Elliptic Functions	27
2.6 Gauss: Theta Functions and Modular Forms	32
2.7 Exercises	36
3 Abel and Jacobi on Elliptic Functions	42
3.1 Preliminary Remarks	42
3.2 Jacobi on Transformations of Orders 3 and 5	54
3.3 The Jacobi Elliptic Functions	60
3.4 Transformations of Order n and Infinite Products	63
3.5 Jacobi's Transformation Formulas	66
3.6 Equivalent Forms of the Transformation Formulas	70
3.7 The First and Second Transformations	71
3.8 Complementary Transformations	72
3.9 Jacobi's First Supplementary Transformation	74
3.10 Jacobi's Infinite Products for Elliptic Functions	75
3.11 Jacobi's Theory of Theta Functions	80
3.12 Jacobi's Triple Product Identity	86
3.13 Modular Equations and Transformation Theory	89
3.14 Exercises	90

4	Eisenstein and Hurwitz	94
4.1	Preliminary Remarks	94
4.2	Eisenstein's Theory of Trigonometric Functions	101
4.3	Eisenstein's Derivation of the Addition Formula	105
4.4	Eisenstein's Theory of Elliptic Functions	106
4.5	Differential Equations for Elliptic Functions	109
4.6	The Addition Theorem for the Elliptic Function	113
4.7	Eisenstein's Double Product	115
4.8	Elliptic Functions in Terms of the ϕ Function	116
4.9	Connection of ϕ with Theta Functions	117
4.10	Hurwitz's Fourier Series for Modular Forms	123
4.11	Hurwitz's Proof That $\Delta(\omega)$ Is a Modular Form	126
4.12	Hurwitz's Proof of Eisenstein's Result	128
4.13	Kronecker's Proof of Eisenstein's Result	129
4.14	Exercises	130
5	Hermite's Transformation of Theta Functions	132
5.1	Preliminary Remarks	132
5.2	Hermite's Proof of the Transformation Formula	138
5.3	Smith on Jacobi's Formula for the Product of Four Theta Functions	141
5.4	Exercises	147
6	Complex Variables and Elliptic Functions	149
6.1	Historical Remarks on the Roots of Unity	149
6.2	Simpson and the <i>Ladies Diary</i>	161
6.3	Development of Complex Variables Theory	164
6.4	Hermite: Complex Analysis in Elliptic Functions	172
6.5	Riemann: Meaning of the Elliptic Integral	176
6.6	Weierstrass's Rigorization	182
6.7	The Phragmén-Lindelöf Theorem	184
7	Hypergeometric Functions	188
7.1	Preliminary Remarks	188
7.2	Stirling	189
7.3	Euler and the Hypergeometric Equation	191
7.4	Pfaff's Transformation	192
7.5	Gauss and Quadratic Transformations	193
7.6	Kummer on the Hypergeometric Equation	196
7.7	Riemann and the Schwarzian Derivative	198
7.8	Riemann and the Triangle Functions	201
7.9	The Ratio of the Periods $\frac{K'}{K}$ as a Conformal Map	202
7.10	Schwarz: Hypergeometric Equation with Algebraic Solutions	207
7.11	Exercises	210
8	Dedekind's Paper on Modular Functions	212
8.1	Preliminary Remarks	212
8.2	Dedekind's Approach	216

<i>Contents</i>		vii
8.3	The Fundamental Domain for $SL_2(\mathbb{Z})$	219
8.4	Tessellation of the Upper Half-plane	222
8.5	Dedekind's Valency Function	222
8.6	Branch Points	223
8.7	Differential Equations	225
8.8	Dedekind's η Function	228
8.9	The Uniqueness of k^2	234
8.10	The Connection of η with Theta Functions	234
8.11	Hurwitz's Infinite Product for $\eta(\omega)$	235
8.12	Algebraic Relations among Modular Forms	236
8.13	The Modular Equation	238
8.14	Singular Moduli and Quadratic Forms	243
8.15	Exercises	249
9	The η Function and Dedekind Sums	251
9.1	Preliminary Remarks	251
9.2	Riemann's Notes	258
9.3	Dedekind Sums in Terms of a Periodic Function	264
9.4	Rademacher	269
9.5	Exercises	274
10	Modular Forms and Invariant Theory	276
10.1	Preliminary Remarks	276
10.2	The Early Theory of Invariants	279
10.3	Cayley's Proof of a Result of Abel	285
10.4	Reduction of an Elliptic Integral to Riemann's Normal Form	287
10.5	The Weierstrass Normal Form	289
10.6	Proof of the Infinite Product for Δ	291
10.7	The Multiplier in Terms of $\sqrt[12]{\Delta}$	293
11	The Modular and Multiplier Equations	295
11.1	Preliminary Remarks	295
11.2	Jacobi's Multiplier Equation	303
11.3	Sohnke's Paper on Modular Equations	304
11.4	Brioschi on Jacobi's Multiplier Equation	314
11.5	Joubert on the Multiplier Equation	317
11.6	Kiepert and Klein on the Multiplier Equation	320
11.7	Hurwitz: Roots of the Multiplier Equation	326
11.8	Exercises	332
12	The Theory of Modular Forms as Reworked by Hurwitz	334
12.1	Preliminary Remarks	334
12.2	The Fundamental Domain	335
12.3	An Infinite Product as a Modular Form	336
12.4	The J -Function	339
12.5	An Application to the Theory of Elliptic Functions	342

13	Ramanujan's Euler Products and Modular Forms	344
	13.1 Preliminary Remarks	344
	13.2 Ramanujan's τ Function	348
	13.3 Ramanujan: Product Formula for Δ	350
	13.4 Proof of Identity (13.2)	353
	13.5 The Arithmetic Function $\tau(n)$	356
	13.6 Mordell on Euler Products	362
	13.7 Exercises	367
14	Dirichlet Series and Modular Forms	371
	14.1 Preliminary Remarks	371
	14.2 Functional Equations for Dirichlet Series	373
	14.3 Theta Series in Two Variables	380
	14.4 Exercises	382
15	Sums of Squares	384
	15.1 Preliminary Remarks	384
	15.2 Jacobi's Elliptic Functions Approach	393
	15.3 Glaisher	394
	15.4 Ramanujan's Arithmetical Functions	397
	15.5 Mordell: Spaces of Modular Forms	400
	15.6 Hardy's Singular Series	405
	15.7 Hecke's Solution to the Sums of Squares Problem	410
	15.8 Exercises	424
16	The Hecke Operators	426
	16.1 Preliminary Remarks	426
	16.2 The Hecke Operators $T(n)$	428
	16.3 The Operators $T(n)$ in Terms of Matrices $\lambda(n)$	434
	16.4 Euler Products	438
	16.5 Eigenfunctions of the Hecke Operators	439
	16.6 The Petersson Inner Product	442
	16.7 Exercises	444
	Appendix: Translation of Hurwitz's Paper of 1904	445
	§1. Equivalent Quantities	445
	§2. The Modular Forms $G_n(\omega_1, \omega_2)$	448
	§3. The Representation of the Function G_n by Power Series	452
	§4. The Modular Form $\Delta(\omega_1, \omega_2)$	454
	§5. The Modular Function $J(\omega)$	455
	§6. Applications to the Theory of Elliptic Functions	460
	<i>Bibliography</i>	463
	<i>Index</i>	471