Elliptic and Modular Functions from Gauss to Dedekind to Hecke

This book presents the fundamental results of modular function theory as developed during the nineteenth and early twentieth centuries, focussing particularly on those interesting methods and techniques that appear to have been overlooked or are not generally well known. Of particular note are Jacobi's derivation of the infinite products for his elliptic functions based on his transformation theory; his first proof of the triple product identity; Hermite's derivation of elliptic functions, establishing the conditions for the ratio of two periodic entire functions to be doubly periodic; Mordell's proof of Ramanujan's conjectures on the Euler products of certain modular forms, based on the work of Hurwitz, Kiepert and Klein on the multiplier equation; and Hecke's work on the representation of integers as sums of squares using Dirichlet series of signature $(2, k, 1)$.

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Preface

Modular function theory has developed a great deal during the past two hundred years, and its foundations have been reworked several times. During the course of this process, some earlier approaches and methods have naturally fallen into disuse or obscurity. The purpose of this work is to present the fundamental results of modular function theory as developed during the nineteenth and early twentieth centuries, focusing particularly on those interesting methods and techniques that appear to have been overlooked or are not generally well known. While the study of these methods has intrinsic value, it may also be useful to contemporary researchers in modular function theory. The study of nineteenth-century mathematics is often believed to be quite cumbersome, but a little familiarity with its mode of presentation and context readily dissolves much of this obstacle. To illustrate this point, I have remained very close to the original expositions and have provided some details about the creators of modular function theory. It is my hope that substantial portions of this book will be accessible to beginning graduate students, and to undergraduates with a solid knowledge of complex analysis.

The triple product identity is the expression of a theta series as an infinite product, and by 1808 this identity was known to Gauss; Chapter 2 is devoted to Gauss's work on theta functions. A special case of the triple product identity, known as the pentagonal number theorem, was conjectured by Euler in 1741 and proved by him in 1750. This theorem is so named because the powers of the variable in the series are pentagonal numbers. Euler proved this result by a very interesting and fruitful method, and Gauss, perhaps around 1796–97, showed that the method could also be applied to the situation in which the powers were squares or triangular numbers. This work of Gauss has received hardly any attention. In 1882, Cayley applied this method to reprove an important identity of Sylvester. Again, in 1919, Ramanujan employed it to prove the Rogers-Ramanujan identity. More recently, Andrews has shown that Euler's method can be applied to verify several significant q-series identities.

Jacobi's fundamental results on infinite products for the Jacobi elliptic functions are discussed in Chapter 3. These infinite products may be obtained in several ways, one due to Abel; most expositions use the method of theta functions, found by Jacobi. However, we present Jacobi's first method, given in his *Fundamenta Nova* of 1829. This method employs the transformation of elliptic functions, a method that has hardly appeared in textbooks since 1900. We also present Jacobi's three proofs of the triple product identity, especially Jacobi's little-known and ingenious first proof.

Eisenstein's method of constructing elliptic and modular functions and his proofs of important basic results have been given in Chapter 4, closely following Eisenstein's 1847 monograph in some detail. We remark that Weil's excellent book[1] gives a concise presentation of this material. Chapter 4 also contains Hurwitz's first 1881 proof of the fact that Weierstrass's discriminant function $\Delta(\omega)$ is expressible as an infinite product in $q = e^{i\pi\omega}$, Im $\omega > 0$. Hurwitz's elegant and more familiar second proof is presented in Chapter 12.

Hermite's noteworthy and useful 1858 work on the transformation of theta functions, dealt with in Chapter 5, led him to his most significant contributions to modular function theory. Hermite employed a powerful new theta functions notation, depending upon two indices. We also discuss Smith's 1866 paper, in which he used this helpful notation to prove Jacobi's formula for the product of four theta functions, a formula from which many other results in elliptic and theta function theory can be easily derived. Jacobi briefly mentioned this formula without any details in a letter to Hermite, included by Jacobi in his 1847 *Opuscula Mathematica*. He wrote that he had derived this formula in his lectures on theta series, delivered in the mid-1830s. These lectures were not available at the time Smith read the *Opuscula*, though they were later published by Borchardt. These lectures have been treated in textbooks, including Stalker's outstanding 1998 book,[2] but Smith's work, initiated by Jacobi's passing remark, has received little attention.

After the initial work by Abel and Jacobi on elliptic and modular functions, mathematicians realized that further advances in this theory required the use of the then-novel methods of complex analysis; we discuss these matters in Chapter 6. Also included is a treatment of the much earlier work of Cotes, of about 1715, and de Moivre, of the 1720s, on the factorization of $x^n - a^n$, and Simpson's 1759 dissection of an infinite series, since these results were applied by Sohnke in 1837 and others to determine the modular equation. Hermite's method of 1848 for construction of elliptic functions, also included in Chapter 6, is a topic now fallen into some obscurity. He started with the observation that a periodic entire function could not have a second period without becoming a constant. He then considered the ratio of two periodic entire functions, each with the same period, and enquired as to the conditions under which this function could have a second period. A particular case of his solution produced the Jacobi elliptic functions as the ratios of theta functions.

Hypergeometric functions, the topic of Chapter 7, encompass many important functions as special or limiting cases. As such, they were a major focus of nineteenth-century mathematical research, on which Gauss, Kummer, and Riemann wrote important papers. Gauss's early interest in hypergeometric functions was aroused by his observation that the complete elliptic integral K could be expressed as a hypergeometric function of k^2. Then Riemann, in his 1856–57 lectures on hypergeometric functions, was able to express them as contour integrals; this insight led him to a proof that the

[1] Weil (1976). [2] Stalker (1998).

modulus k^2 was invariant with respect to the principal congruence subgroup of level 2. Although it is a striking example of the way an integral changes as the contour moves around singular points, Riemann's fascinating proof of this theorem does not seem to have made its way into our textbooks or discussions.

Dedekind's impressive effort to provide a foundation for the theory of modular functions independent of elliptic functions is the main topic of Chapters 8 and 9. He described the fundamental domain of the modular group and defined the modular invariant by means of a conformal mapping of the fundamental domain onto the complex plane. He denoted this mapping function by v; a year later, Klein named it J, so that it has since been called Klein's J invariant. Dedekind also introduced his famous η function, defined as

$$\eta(\omega) = cv^{-\frac{1}{6}}(\omega)\,(1-v)^{-\frac{1}{8}}\left(\frac{dv}{d\,\omega}\right)^{\frac{1}{4}},$$

where c denotes a constant. He then defined the elliptic modular functions k^2, K, k'^2, and K' in terms of the η function. Dedekind regretted the fact that, in the end, he was forced to employ theta functions to show that the η function could be expressed as an infinite product. Hurwitz used Eisenstein series to overcome this difficulty.

Modern mathematicians define the η function immediately as an infinite product, and, in fact, Dedekind himself had taken this step in his comments on some fragmentary results of Riemann. In these remarks, Dedekind proved, without the use of integrals, that the modular transformation of the η function was expressible in terms of a finite arithmetical sum. Rademacher called this a Dedekind sum and developed its properties independent of its η function origins. For example, he proved that the Dedekind sum could be expressed as a finite sum of values of the cotangent function. Chapter 9 includes Rademacher's first proof of this result, utilizing integrals; the proof given by Rademacher in his well-known book on Dedekind sums avoids this device.

Weierstrass and Klein elucidated the relationship of algebraic invariants and modular forms, the topic of Chapter 10. Klein gave the modern definition of the J invariant:

$$J(\omega) = \frac{g_2^3(\omega)}{\Delta(\omega)}.$$

He then showed that his invariant-theoretic approach allowed him to apply Jacobi's results to obtain the infinite product for Δ and the Lambert series expansion for g_2.

Chapter 11 discusses modular and multiplier equations, connecting up Jacobi's work with that of Dedekind, Klein, Kiepert, and Hurwitz, and leading to Mordell's resolution of Ramanujan's conjectures. Even before he discovered elliptic functions, Jacobi used a trial-and-error method to find modular equations of orders 3 and 5. Upon discovering elliptic functions, Jacobi was able to establish the $p+1$ roots of the modular equation of prime order p. A few years later, Sohnke made use of these roots to construct modular equations of orders 7, 11, 13, 17, and 19. Jacobi also found the fifth-order multiplier equation. Three decades later, Joubert showed that methods analogous to those of Sohnke could be employed to develop multiplier equations of small prime order. Chapter 11 gives an expanded treatment of the methods of Sohnke and Joubert, who were mentioned briefly by Borwein and Borwein.[3]

Kiepert and Klein saw that $\sqrt[12]{\Delta(\omega)}$ could be interpreted as a multiplier in the transformation of an elliptic integral. Klein came to this conclusion by way of his invariant-theoretic approach, whereas Kiepert arrived at this idea through his use of Weierstrass elliptic functions to solve algebraic equations of the fifth degree. As Kiepert's teacher, Weierstrass had suggested that such a method might lead to a solution of a general equation of the fifth degree without its reduction to an equation of a special form. Although this suggestion did not pan out, it led Kiepert to the consideration of the transformation equation satisfied by $\sqrt[12]{\Delta(p\,\omega)}$. Klein gave the roots of the general equation of order p up to a factor of a twelfth root of unity. His student, Hurwitz, determined these twelfth roots exactly in his 1881 thesis. It was this work of Hurwitz that Mordell employed to resolve the conjectures of Ramanujan on Euler products associated with modular forms.

The ideas of Hurwitz's innovative 1904 paper are well known and have appeared in many textbooks, perhaps because Serre discussed them in a 1957 seminar and soon thereafter presented them in his *Cours d'arithmétique*. Previous chapters concentrated primarily on methods now little known; by contrast, Chapter 12 presents familiar results and methods within modular function theory, but as Hurwitz first thought them out. Such mathematical results, as originally conceived and presented even one hundred or more years ago, may well reveal completely new or lost insights and avenues. A translation of Hurwitz's original 1904 paper is included as an appendix to this book, by which the reader may see that the ideas contained in old mathematical papers are often very accessible.

Ramanujan may be seen as the founder of an essential aspect of the theory of modular forms: the theory of Euler products, the topic of Chapter 13. It is generally well known that Ramanujan conjectured that the Dirichlet series associated with $\Delta(\omega)$ had an Euler product. However, his work in this area was much more extensive, including his statement that similar Euler products must exist for $\sqrt{\Delta},\ g_2\,\Delta,\ g_3\,\Delta,\ g_2^2\,\Delta,\ g_2\,g_3\,\Delta,\ g_2^2\,g_3\,\Delta$. Moreover, he correctly surmised that linear combinations of modular forms of a given level had Euler products even when the original modular forms did not. This led Birch to remark in 1975[4] that Ramanujan's insight into the arithmetic of modular forms was greater than initially realized. It is often mentioned that Mordell proved Ramanujan's conjecture on $\Delta(\omega)$, usually in the context of Hecke's work. But again, as we discuss in Chapter 13, Mordell also proved Ramanujan's conjectures that other related modular forms had Euler products.

Hecke's work on Dirichlet series and modular forms, that is, series and functions of signature $\{\lambda, k, \gamma\}$, has been thoroughly treated in Berndt and Knopp's recent book.[5] Consequently, I have confined Chapter 14 to the background to and foundation for Hecke's work and a key result of Hecke on the dimension of the space of functions of signature $\{2, k, 1\}$. Chapter 15 is devoted to Hecke's little-known work on representations of integers as sums of squares, with additional mention of Glaisher's contributions in this area. In lectures at Princeton and Michigan in 1938, Hecke showed how

[3] Borwein and Borwein (1987). [4] Birch (1975). [5] Berndt and Knopp (2008).

Dirichlet series of signature {2, k, 1}, with k a positive integer, could be applied to the problem of the number of representations of an integer as a sum of an even number of squares. Hecke is not mentioned in Grosswald's encyclopedic work of 1983[6] on sums of squares; Berndt and Knopp mention this work of Hecke without too much detail.

Hecke operators on modular forms for the full modular group is the topic of Chapter 16, with the exposition closely following Hecke's own treatment of the topic. Thus, this chapter completes the narrative of the previous three chapters on the connection between modular forms and Dirichlet series. Because Hecke was unaware that his operators were Hermitian, his work is somewhat labored and incomplete; still, it is of interest to note how much he could do under this constraint. Hecke's student Petersson showed that the operators were Hermitian, thus greatly simplifying the theory.

In this book we can see in some detail that a study of the relevant works of past centuries can greatly expand one's mathematical perspective and tool box. In addition, as one studies the development of the theory of modular functions, one encounters acknowledged masters, such as Gauss, Dedekind, Hecke, and others. Yet neglecting the work of lesser-known mathematicians, such as Joubert, Kiepert, and Sohnke, would be regrettable, since they added significant insights and advances. Clearly, the rate of mathematical output has increased at least thirty- or forty-fold since the early twentieth century. Thus, there is a relatively small body of work dating from the nineteenth or earlier centuries, and of this body, very few papers are of lasting interest within a given topic. Reading those few papers is not excessively demanding, and the benefits of so doing, especially for one's teaching, far outweigh any inconvenience. With increasing electronic availability, study of the old works has been made even easier.

My next volume will focus on the developments in modular forms after Hecke's 1935–37 papers, discussing connections with topics such as quadratic forms, elliptic curves, and Ramanujan's conjectures on partitions. There exist many introductory works on such topics, so I will attempt to elucidate important and interesting aspects of the more advanced topics within a manageable number of pages.

I first thank my wife for typesetting and editing this work and for her valuable help in translating; note here that unattributed translations are mine. I am indebted to NFN Kalyan for creating the wonderful portrait of Gauss for the cover. Kalyan's work is constructed of ten layers of etched glass, illuminated by colored LED light. I also owe a great debt to Kieran Donaghue for his expert assistance in German translations. I am grateful for Bruce Atwood's and Zhitai Li's skillful construction and corrections of diagrams. Cindy Cooley and Chris Nelson provided indispensable help with library materials; Sarah Arnsmeier assisted me with secretarial work. Thanks to Paul Campbell for guidance on making the bibliography. Finally, I thank Ann Davies and Beloit College for supporting me during the process of preparing this book.

[6] Grosswald (1983).