1

Four Small Examples

We recall basic notions related to automorphic forms on some simple arithmetic quotients, including the archetypical quotient $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ of the complex upper half-plane $\mathfrak{H}$ and the related $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$. To put this in a somewhat larger context,[1] we consider parallel examples $\Gamma\backslash X$ and $\Gamma\backslash G$ for a few other groups G, discrete subgroups Γ, and spaces $X \approx G/K$ for compact subgroups K of G. The

[1] In slightly more sophisticated terms inessential to this discussion: the four examples G immediately considered are *real-rank one* semi-simple Lie groups, and the discrete subgroups Γ are *unicuspidal* in the sense that $\Gamma\backslash G/K$ is reasonably compactified by adding just a single *cusp*, where K is a (maximal) compact subgroup of G. That is, the *reduction theory* of $\Gamma\backslash G$ is especially simple in these four cases. Examples with larger real rank, such as GL_n with $n \geq 3$, are considered later.

other three examples share several of the features of $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, and $X = \mathfrak{H} \approx G/K$ with $K = SO_2(\mathbb{R})$, allowing simultaneous treatment.

For many reasons, even if we are only interested in harmonic analysis on quotients $\Gamma\backslash X$, it is necessary to consider spaces of functions on the *overlying* spaces $\Gamma\backslash G$, on which G acts by right translations, with a corresponding translation action on functions.

Some basic discussions *not* specific to the four examples are postponed, such as determination of invariant Laplacians in coordinates, self-adjointness properties of invariant Laplacians, proof of the formula for the left G-invariant measure on $X = G/K$, unwinding properties of integrals and sums, continuity of the action of G on *test functions* on $\Gamma\backslash G$, density of test functions in $L^2(\Gamma\backslash X)$, vector-valued integrals, holomorphic vector-valued functions, and other generalities.

We also postpone the relatively *specific* proofs of the major theorems *stated* in the final sections of this chapter, concerning the spectral decomposition of automorphic forms, meromorphic continuation of Eisenstein series, and the theory of the constant term. Those proofs make pointed use of finer details from the more sophisticated analysis.

1.1 Groups $G = SL_2(\mathbb{R})$, $SL_2(\mathbb{C})$, $Sp^*_{1,1}$, and $SL_2(\mathbb{H})$

These four groups share some convenient simplifying features, which we will exploit. The first two examples G are easy to describe:

$$G = \begin{cases} SL_2(\mathbb{R}) = \text{a } \textit{special linear} \text{ group over } \mathbb{R} \\ \qquad\qquad = \text{two-by-two determinant-1 real matrices} \\ SL_2(\mathbb{C}) = \text{a } \textit{special linear} \text{ group over } \mathbb{C} \\ \qquad\qquad = \text{two-by-two determinant-1 complex matrices} \end{cases}$$

We will have occasion to use the *general linear* groups $GL_2(R)$ of 2-by-2 invertible matrices with entries in a ring R. Our other two example groups are conveniently described in terms of the Hamiltonian *quaternions* $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, with the usual relations

$$i^2 = j^2 = k^2 = -1 \qquad ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j$$

The quaternion conjugation is $\overline{\alpha} = \overline{a + bi + cj + dk} = a - bi - cj - dk$ for $\alpha = a + bi + cj + dk$, the *norm* is $N\alpha = \alpha \cdot \overline{\alpha}$, and $|\alpha| = (N\alpha)^{\frac{1}{2}}$. $\mathbb{H}$ can be modeled in two-by-two complex matrices by

$$\rho(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

with $\det \rho(\alpha) = N\alpha$. For a quaternion matrix g, let g^* be the transpose of the entry-wise conjugate:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} \qquad (\text{for } \alpha, \beta, \gamma, \delta \in \mathbb{H})$$

The third example group is a kind of *symplectic group*: letting $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, this group is

$$G = Sp^*_{1,1} = \{g \in GL_2(\mathbb{H}) : g^* S g = S\}$$

The fourth example is a *special linear* group $G = SL_2(\mathbb{H})$. In the latter, SL_2 is more convenient than GL_2, having a smaller center. However, because $\mathbb{H}$ is not commutative, the notion of *determinant* is problemmatic. One way to skirt the issue is to imbed $r : GL_2(\mathbb{H}) \to GL_4(\mathbb{C})$: with quaternions $\alpha, \beta, \gamma, \delta$,

$$r\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \rho(\alpha) & \rho(\beta) \\ \rho(\gamma) & \rho(\delta) \end{pmatrix}$$

identified with a 4-by-4 complex matrix, using the map ρ of $\mathbb{H}$ to 2-by-2 complex matrices, and require that the *image* in $GL_4(\mathbb{C})$ be in the subgroup $SL_4(\mathbb{C})$ where determinant is 1:

$$SL_2(\mathbb{H}) = \{g \in GL_2(\mathbb{H}) : r(g) \in SL_4(\mathbb{C})\}$$

Standard subgroups of any of these groups G are

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \quad N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\} \quad M = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\}$$
$$A^+ = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0\}$$

The *Levi-Malcev decomposition* $P = NM$ is elementary to check. By direct computation from the defining relations of the groups, one finds

$$M = \begin{cases} \{\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} : m \in \mathbb{R}^\times\} & (\text{for } G = SL_2(\mathbb{R})) \\ \{\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} : m \in \mathbb{C}^\times\} & (\text{for } G = SL_2(\mathbb{C})) \\ \{\begin{pmatrix} m & 0 \\ 0 & \overline{m}^{-1} \end{pmatrix} : m \in \mathbb{H}^\times\} & (\text{for } G = Sp^*_{1,1}) \\ \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : N(ad) = 1,\ a, d \in \mathbb{H}^\times\} & (\text{for } G = SL_2(\mathbb{H})) \end{cases}$$

and

$$N = \begin{cases} \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\} & (\text{for } G = SL_2(\mathbb{R})) \\ \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{C}\} & (\text{for } G = SL_2(\mathbb{C})) \\ \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{H},\ x + \bar{x} = 0\} & (\text{for } G = Sp^*_{1,1}) \\ \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{H}\} & (\text{for } G = SL_2(\mathbb{H})) \end{cases}$$

The subgroup P is the *standard (proper) parabolic*, N is its *unipotent radical*, M is the standard *Levi-Malcev component*, and A^+ is the *standard split component*. We will use these (standard) names without elaborating on their history or their connotations.

In these examples, the (spherical) Bruhat decomposition is

$$G = \bigsqcup_{w=1,w_o} PwP = P \sqcup Pw_oP = P \sqcup Pw_oN$$

where $w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, with the last equality following because w_o normalizes M:

$$Pw_oP = Pw_oMN = P(w_oMw_o^{-1})w_oN = Pw_oN$$

The element w_o is the *long Weyl element*. The *small (Bruhat) cell* is P itself, and the *big (Bruhat) cell* is Pw_oP. The (spherical, geometric) Weyl group is $\{1, w_o\}$. It is a group modulo the center of G. The proof of the Bruhat decomposition is straightforward: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P$ if and only if $c = 0$. Otherwise, $c \neq 0$, and we try to find $p \in P$ and $n \in N$ such that $g = pw_on$. To simplify, since $c \neq 0$, it is invertible, so, in a form applicable to all four cases, we can left multiply by $\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \in M$ to make $c = 1$ without loss of generality. Then try to solve

$$\begin{aligned} \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} = g = pw_on &= \begin{pmatrix} p_{11} & p_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n_{12} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} p_{12} & p_{12}n_{12} - p_{11} \\ 1 & n_{12} \end{pmatrix} \end{aligned}$$

From the lower right entry, apparently $n_{12} = d$. For the case $G = Sp^*_{1,1}$ the additional condition must be checked, as follows. Observe that inverting $g^*Sg = S$ gives $g^{-1}S^{-1}(g^*)^{-1} = S^{-1}$, and then $S = gSg^*$. In particular, this gives a relation between the c, d entries of g:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S = gSg^* = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} * & \overline{c} \\ * & \overline{d} \end{pmatrix}$$
$$= \begin{pmatrix} * & * \\ * & c\overline{d} + d\overline{c} \end{pmatrix}$$

For $c = 1$, this gives $d + \overline{d} = 0$, which is the condition for $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \in N$ in that case. Thus, in all cases, right multiplying g by $\begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} \in N$ makes $d = 0$, without loss of generality. Thus, it suffices to solve

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = g = pw_o = \begin{pmatrix} p_{11} & p_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{12} & -p_{11} \\ 1 & 0 \end{pmatrix}$$

That is,

$$gw_o^{-1} = \begin{pmatrix} -b & a \\ 0 & 1 \end{pmatrix} = p$$

Since $g \in G$, the entries a, b satisfy whatever relations G requires, and $p \in G$. This proves the Bruhat decomposition.

1.2 Compact Subgroups $K \subset G$, Cartan Decompositions

We describe the standard maximal[2] compact subgroups $K \subset G$ for the four examples G. With $\mathbb{H}^1$ the quaternions of norm 1, in a notation consistent with that for $Sp^*_{1,1}$, write

$$Sp^*_1 = \{g \in GL_1(\mathbb{H}) : g^*g = 1\} = \{g \in \mathbb{H}^\times : \overline{g}g = 1\} = \mathbb{H}^1$$

Letting 1_2 be the two-by-two identity matrix, the four maximal compact subgroups are

$$K = \begin{cases} SO_2(\mathbb{R}) & = & \{g \in SL_2(\mathbb{R}) : g^\top g = 1_2\} & (\text{for } G = SL_2(\mathbb{R})) \\ SU_2 & = & \{g \in SL_2(\mathbb{C}) : g^*g = 1_2\} & (\text{for } G = SL_2(\mathbb{C})) \\ Sp^*_1 \times Sp^*_1 & = & \mathbb{H}^1 \times \mathbb{H}^1 & (\text{for } G = Sp^*_{1,1}) \\ Sp^*_2 & = & \{g \in GL_2(\mathbb{H}) : g^*g = 1_2\} & (\text{for } G = SL_2(\mathbb{H})) \end{cases}$$

[2] The maximality of each of these subgroups K among all compact subgroups in the corresponding G is *not* obvious but is not used in the sequel.

In all four cases, the indicated groups *are* compact. Verification of the compactness of the first three is straightforward because their defining equations present them as spheres or products of spheres. Verification that Sp_2^* is compact and is a subgroup of $SL_2(\mathbb{H})$ merits discussion. For the fourth, observe that the defining condition

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |c|^2 & \overline{a}b + \overline{c}d \\ \overline{b}a + \overline{d}c & |b|^2 + |d|^2 \end{pmatrix}$$

makes Sp_2^* a closed subset of a product of two seven-spheres, $|a|^2 + |c|^2 = 1$ and $|b|^2 + |d|^2 = 1$, thus, compact. Further, Sp_2^* lies inside $SL_2(\mathbb{H})$ rather than merely $GL_2(\mathbb{H})$. For the moment, we will prove a slightly weaker property, that the relevant determinant is ± 1. Use the feature

$$\rho(\overline{\alpha}) = \varepsilon \rho(\alpha)^\top \varepsilon^{-1} \qquad \left(\text{where } \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ for } \alpha \in \mathbb{H}\right)$$

of the imbedding ρ of $\mathbb{H}$ in 2-by-2 complex matrices, and again let

$$r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \rho(a) & \rho(b) \\ \rho(c) & \rho(d) \end{pmatrix} \qquad (\text{for } a, b, c, d \in \mathbb{H})$$

viewed as mapping to 4-by-4 complex matrices. Then $r(g^*) = J \cdot r(g) \cdot J^{-1}$, where

$$J = \begin{pmatrix} \varepsilon & \\ & \varepsilon \end{pmatrix} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

and $g \in GL_2(\mathbb{H})$. Thus, for $g^* g = 1_2 \in GL_2(\mathbb{H})$,

$$1_4 = r(1_2) = r(g^* g) = r(g^*) \cdot r(g) = J \cdot r(g)^\top \cdot J^{-1} \cdot r(g)$$

In other words, $r(g)^\top J r(g) = J$.[3] Taking determinants shows $\det r(g)^2 = 1$, so $\det r(g) = \pm 1$. Thus, g in the *connected component* of Sp_2^* containing 1 has $\det r(g) = 1$.

The copy K of $Sp_1^* \times Sp_1^*$ inside $Sp_{1,1}^*$ is not immediately visible in these coordinates, which were chosen to make the *parabolic* P visible. That is, defining $Sp_{1,1}^*$ as the isometry group of the quaternion Hermitian form S obscures the nature of the (maximal) compact K. Changing coordinates by replacing S by

$$S' = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} S \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^\top = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

[3] Thus, $r(g)$ is inside a *symplectic group* denoted $Sp_4(\mathbb{C})$ or $Sp_2(\mathbb{C})$, depending on convention.

gives

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} Sp^*_{1,1} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \{g \in GL_2(\mathbb{H}) : g^* S' g = S'\}$$

and makes the two copies of Sp^*_1 visible on the diagonal:

$$\left\{ k = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} : k^* S' k = S' \right\} = \left\{ k = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} : \mu, \nu \in \mathbb{H}^1 \right\}$$

That is,

$$\begin{aligned} K &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} : \mu, \nu \in \mathbb{H}^1 \right\} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} \frac{\mu+\nu}{2} & \frac{-\mu+\nu}{2} \\ \frac{-\mu+\nu}{2} & \frac{\mu+\nu}{2} \end{pmatrix} : \mu, \nu \in \mathbb{H}^1 \right\} \end{aligned}$$

[1.2.1] Claim: $K \cap P = K \cap M$, and

$$K \cap M = \begin{cases} \pm 1_2 & (\text{for } G = SL_2(\mathbb{R})) \\ \{\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} : \mu \in \mathbb{C}^\times,\ |\mu| = 1\} & (\text{for } G = SL_2(\mathbb{C})) \\ \{\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} : \mu \in \mathbb{H}^1\} & (\text{for } G = Sp^*_{1,1}) \\ \{\begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} : \mu,\ \nu \in \mathbb{H}^1\} & (\text{for } G = SL_2(\mathbb{H})) \end{cases}$$

Proof: In all but the third case, this follows from the description of K. For example, for $G = SL_2(\mathbb{R})$ and $K = SO_2(\mathbb{R})$, take $p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P$ and examine the relation $p^\top p = 1_2$ for p to be in K:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = p^\top p &= \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a^2 & (a+a^{-1})b \\ (a+a^{-1})b & b^2 + a^{-2} \end{pmatrix} \end{aligned}$$

From the upper-left entry, $a = \pm 1$. From the off-diagonal entries, $b = 0$. The arguments for $SL_2(\mathbb{C})$ and $SL_2(\mathbb{H})$ are similar. For $Sp^*_{1,1}$, comparison to the

coordinates that diagonalize $K \approx Sp_1^* \times Sp_1^*$ gives

$$\begin{aligned}\{\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{H}^1\} &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} K \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \\ &\ni \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & (a^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \tfrac{1}{2} \begin{pmatrix} a + (a^*)^{-1} - b & a - (a^*)^{-1} + b \\ a - (a^*)^{-1} - b & a + (a^*)^{-1} + b \end{pmatrix}\end{aligned}$$

For example, adding the elements of the bottom row gives $a = k_2 \in \mathbb{H}^1$ and also $(a^*)^{-1} = a$. From either off-diagonal entry, $b = 0$. ///

In all four cases, the same discussion gives $M = A^+ \cdot (P \cap K) = A^+ \cdot (M \cap K)$.

The following will be essential in [7.1]:

[1.2.2] Claim: *(Cartan decomposition)* $G = KA^+K$.

Proof: First, treat $G = SL_2(\mathbb{R})$. Prove that every $g \in G$ can be written as $g = sk$ with $s^\top = s$ and s positive-definite. To find such s, assume for the moment that it exists, and consider

$$g \cdot g^\top = (sk) \cdot (sk)^\top = sk \cdot k^{-1}s = s^2$$

Certainly $gg^\top$ is symmetric and positive-definite, so having a positive-definite symmetric square root of positive-definite symmetric t would produce s. Such t gives a positive, symmetric operator on $\mathbb{R}^2$, which by the spectral theorem has an orthonormal basis of eigenvalues. That is, there is $h \in K$ such that $hth^\top = \delta$ is diagonal, necessarily with positive diagonal entries. With $\delta^{\frac{1}{2}}$ being the positive diagonal square root of δ,

$$(h^\top \delta^{\frac{1}{2}} h)^2 = h^\top \delta^{\frac{1}{2}} h \cdot h^\top \delta^{\frac{1}{2}} h = h^\top \delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}} h = h^\top \delta h = t$$

Thus, take $s = h^\top \delta^{\frac{1}{2}} h$, and every $g \in G$ can be written as $g = ks$. Indeed, we have more:

$$g = ks = k \cdot h^\top \delta^{\frac{1}{2}} h = (k \cdot h^\top) \cdot \delta^{\frac{1}{2}} \cdot h \in K \cdot A^+ \cdot K$$

giving the claim in this case. The cases of $G = SL_2(\mathbb{C})$ is similar, using $g = sk$ with $s = s^*$ Hermitian positive-definite and $k^* = k^{-1} \in K$, invoking the spectral theorem for Hermitian positive-definite operators. The same argument succeeds for $G = SL_2(\mathbb{H})$ with quaternion conjugation replacing complex, with a

suitably adapted spectral theorem for $s \in GL_2(\mathbb{H})$ with $s^* = s$ and x^*sc real and positive for all nonzero 2-by-1 quaternion matrices x.[4]

The case of $G = Sp^*_{1,1}$ essentially reduces to the case of $SL_2(\mathbb{H})$, as follows. Since $g^*Sg = S$, $SgS^{-1} = (g^*)^{-1}$. Anticipating the Cartan decomposition $g = sk$, from $gg^* = ss^* = s^2$, by the quaternionic version of the spectral theorem, there is $k \in Sp^*_2$ such that $k^{-1}gg^*k = \Lambda$ with Λ positive real diagonal. We want to adjust k to be in $Sp^*_{1,1} \cap Sp^*_2$, while preserving the property $k^{-1}gg^*k = \Lambda$. Unless gg^* is *scalar*, the diagonal entries are distinct. By $SgS^{-1} = (g^*)^{-1}$ and $Sg^*S^{-1} = g^{-1}$ for $g \in G$,

$$\begin{aligned}\Lambda^{-1} &= (\Lambda^*)^{-1} = S\Lambda S^{-1} = S(k^{-1}gg^*k)S^{-1} \\ &= (SkS^{-1})^{-1} \cdot Sgg^*S^{-1} \cdot SkS^{-1} = (SkS^{-1})^{-1} \cdot (gg^*)^{-1} \cdot SkS^{-1}\end{aligned}$$

Inverting gives $\Lambda = (SkS^{-1})^{-1} \cdot gg^* \cdot SkS^{-1}$. Also $\Lambda = k^{-1}gg^*k$, so

$$(SkS^{-1}) \cdot \Lambda \cdot (SkS^{-1})^{-1} = gg^* = k \cdot \Lambda \cdot k^{-1}$$

That is, $k^{-1} \cdot SkS^{-1}$ commutes with Λ, and $\delta = k^{-1} \cdot SkS^{-1}$ is at worst diagonal:

$$SkS^{-1} = k \cdot \delta = k \cdot \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

Since $\delta \in Sp^*_2$, $a \cdot \overline{a} = 1$ and $d \cdot \overline{d} = 1$. To preserve $k^{-1}gg^*k = \Lambda$, to adjust k to be in $K = Sp^*_2 \cap Sp^*_{1,1}$, adjust k by diagonal matrices ε in Sp^*_2. The condition for $k\varepsilon$ to be in K is

$$(k \cdot \varepsilon) = ((k\varepsilon)^*)^{-1} = S(k\varepsilon)S^{-1} = SkS^{-1} \cdot S\varepsilon S^{-1} = k \cdot \delta \cdot S\varepsilon S^{-1}$$

so take $\varepsilon = S^{-1}\delta S$. The rest of the argument runs as in the first three cases. ///

1.3 Iwasawa Decomposition $G = PK = NA^+K$

The subgroups P and K are *not* normal in G, so the *Iwasawa decompositions* $G = PK = \{pk : p \in P,\ k \in K\}$ do *not* express G as a product group. Nevertheless, these decompositions are essential.

[1.3.1] Claim: (*Iwasawa decomposition*) $G = PK = NA^+K$. In particular, the map $N \times A^+ \times K \longrightarrow G$ by $n \times a \times k \longrightarrow nak$ is an *injective* set map (and is a diffeomorphism).

[4] In all three of these cases, a Rayleigh-Ritz approach gives a sufficient spectral theorem, as follows. Let F be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $\langle x, y\rangle = y^*x$ for 2-by-1 matrices x, y over F. Let $T : F^2 \to F^2$ be right F-linear and positive Hermitian in the sense that $\langle Tx, x\rangle$ is positive, real for $x \neq 0$. Then x with $\langle x, x\rangle = 1$ maximizing $\langle Tx, x\rangle$ is an eigenvector for T. For nonscalar T, the unit vector y minimizing $\langle Ty, y\rangle$ is an eigenvector for T orthogonal to x. Letting k be the matrix with columns x, y, the conjugated matrix $k^{-1}Tk$ is diagonal.

Proof: For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, in the easy case that $c = 0$, then $g \in P$. In all cases, once we have $g = nm \in P$, we can adjust g on the right by $M \cap K$ to put the Levi component m into A^+.

One approach is to think of right multiplication by K as *rotating* the lower row $(c\ d)$ of $g \in G$ to put it into the form $(0\ *)$ of the lower row of an element of P. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL_2(\mathbb{R})$: right multiplication by the explicit element

$$k = \begin{pmatrix} \dfrac{d}{\sqrt{c^2+d^2}} & \dfrac{c}{\sqrt{c^2+d^2}} \\ \dfrac{-c}{\sqrt{c^2+d^2}} & \dfrac{d}{\sqrt{c^2+d^2}} \end{pmatrix} \in K = SO_2(\mathbb{R})$$

puts $gk \in P$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \dfrac{d}{\sqrt{c^2+d^2}} & \dfrac{c}{\sqrt{c^2+d^2}} \\ \dfrac{-c}{\sqrt{c^2+d^2}} & \dfrac{d}{\sqrt{c^2+d^2}} \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Similarly, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL_2(\mathbb{C})$, right multiplication by

$$k = \begin{pmatrix} \dfrac{d}{\sqrt{|c|^2+|d|^2}} & \dfrac{\overline{c}}{\sqrt{|c|^2+|d|^2}} \\ \dfrac{-c}{\sqrt{|c|^2+|d|^2}} & \dfrac{\overline{d}}{\sqrt{|c|^2+|d|^2}} \end{pmatrix} \in K = SU(2)$$

gives $gk \in P$. Likewise, for $G = SL_2(\mathbb{H})$, nearly the same explicit expression as for $SL_2(\mathbb{C})$ succeeds, with complex conjugation replaced by quaternion conjugation, accommodating the noncommutativity:[5]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \dfrac{c^{-1}d}{\sqrt{1+|c^{-1}d|^2}} & \dfrac{1}{\sqrt{1+|c^{-1}d|^2}} \\ \dfrac{-1}{\sqrt{1+|c^{-1}d|^2}} & \dfrac{\overline{c^{-1}d}}{\sqrt{1+|c^{-1}d|^2}} \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in P$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = Sp^*_{1,1}$, we hope that a matrix k of a similar form lies in $K \approx Sp^*_1 \times Sp^*_1$, and then $gk \in P$. To be sure that the defining relation for

[5] This explicit element lies in the connected component of Sp^*_2 containing 1, so this argument for the Iwasawa decomposition is complete whether or not we have verified that $Sp^*_2 \subset SL_2(\mathbb{H})$.