Cambridge University Press
978-1-107-15250-2 — Quine, New Foundations, and the Philosophy of Set Theory
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Excerpt
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Introduction

Since the late 1960s, there has been widespread agreement among both philosophers and mathematicians that the only viable conception of a set is that of the iterative conception as embodied in Zermelo–Fraenkel (ZF) set theory and its related systems.[1] Indeed, in any discussion, set theory is now treated as practically synonymous with ZF set theory, and the iterative conception is treated as something near essential to the very nature of sets.[2] Other approaches to set theory are generally neglected if not dismissed outright in set-theoretic research. However, this has not always been the situation, as I hope to show in this book by examining W. V. Quine's set theory, New Foundations (NF), and its place in the philosophy of set theory more generally. I should make clear at the outset that what follows is not meant to be an argument in favor of NF over other set theories. Rather, the point will be that work in set theory generally, at least in its current state, should be conducted in the more pluralistic and pragmatic way that I take to be characteristic of what I will identify as the approach to set theory as explication. I describe the contrasting approach, which aims to discover a single correct notion of set, as set theory as conceptual analysis. Let me begin first, though, with some very general history that will be further developed in the main body of the text.

[1] The iterative conception of set had received some attention in the literature on the foundations of set theory, but its first thorough treatment came in George Boolos, "The Iterative Conception of Set," reprinted in his *Logic, Logic, Logic*, ed. Richard Jeffrey (Cambridge, MA: Harvard University Press, 1998), pp. 13–29.

[2] Admittedly, few if any philosophers of set theory talk of capturing the essence of a set; essence has become a fairly unfashionable notion in contemporary philosophy with its drive toward scientific methods and naturalism. Still, it seems to me that the near dogmatic adherence to the iterative conception suggests a commitment to it yielding something like the essence of a set. It is not hard to find examples of the view that sets *must* be like this, where the "this" is filled in by the iterative conception. We will see an example of the sort of view I have in mind in Chapter 5 in discussing Boolos's account of the iterative conception.

Around 1900, in light of the discovery of the set-theoretic paradoxes, the concept of set had to be rethought. Up to that point, this concept had been accepted as basically well understood, taken to be what philosophers and logicians had traditionally meant by the extension of a concept or of a predicate. The realization that not every predicate could determine an extension, on pain of paradox, left set theory at somewhat of a loss in accounting for its fundamental notion. The idea that a set is the extension of a predicate could be captured formally by what has become known as the comprehension principle, "$(\exists x)(\forall y)\ (x \in y \equiv \varphi)$," where x is not free in the formula "φ." But if we let "φ" be "$y \notin y$," for example, we can then very easily derive Bertrand Russell's paradox of the set of all non-self-membered sets. If this set is a member of itself, then it is not a member of itself; and if it is not a member of itself, then it is a member of itself – hence a contradiction.[3] This principle would have to be somehow restricted; but beyond the intuitive thought that a set is the extension of a predicate, which then also involves a commitment to sets to being extensional entities, the tradition did not have much more to go on.

Instead of looking for something inherent in the notion of a set that would yield an appropriately restricted existence principle, logicians and set theorists focused on what they took to be the mathematical value of the theory. In particular, set theory could provide a rigorous foundation for all of ordinary mathematics. The various number systems and operation on them, the notions of a relation and of a function, and even geometry could all be reduced to sets along with the sole relation of membership and logical vocabulary. In addition to this, set theory had to provide an account of the mathematics of the infinite, since Georg Cantor's discovery of an apparently consistent arithmetic of infinite numbers had been the real advance gained for mathematics by way of set theory. Previously, mathematicians had mostly thought any sort of actual infinite to be inherently contradictory. They had operated instead with only potential infinites. In its most basic form, this is the idea that we could always add 1 but that this yielded only another – though larger – finite number. Cantor's development of a rule-governed arithmetic of the infinite in a sense legitimized the infinite for the mathematical community. Hence a successful reworking of set theory was to restrict the comprehension principle enough to prevent the paradoxes but not so much that

[3] This is the simplest of the set-theoretic paradoxes, and it is perhaps for this reason that is the best known among them. There are other paradoxes, of course, which I will discuss further in the text. In particular, there are also Cantor's paradox of the largest cardinal number and the Burali-Forti paradox of the largest ordinal number.

it would lack the power it needed to capture ordinary mathematics as well as the mathematics of the infinite. Such broad criteria left much freedom for choosing how to do this. As we will see, the two main approaches that emerged were Ernst Zermelo's set theory and its related systems and Russell's theory of types. Both were considered to be legitimate approaches to set theory, and both were used widely in foundational studies for mathematics through at least the early 1960s; after that, Zermelo's theory rose to dominance.[4] Neither was considered to capture anything like the essence of a set unless the essence is minimally understood as including some restricted version of the comprehension principle, an axiom of extensionality (sets are identical when they have the same members; this is a distinguishing feature of sets, so much so that it would be hard to consider a theory that lacked it as set theory), and enough strength to capture what made set theory mathematically interesting in the first place.

This period of set theory up through the 1950s, then, was largely a pragmatic undertaking, involving much exploration and experimentation and a certain open-mindedness about how set theory should be done. In addition to the main systems of Russell and Zermelo, many variants of these systems were also studied.[5] There was much interest in whether these different set theories could decide certain fundamental questions about sets that were still open, such as the axiom of choice (AC) or Cantor's continuum hypothesis. There was also a general interest in the comparative study of the various options: How strong were they in comparison to each other, how much mathematics could they prove given their various assumptions, or how much could they tell us about the infinite? Aside from these main systems, there were other sets theories that diverged from them, though they, too, would adhere to the basic criteria

[4] It was dominant at least as a set theory. Russell's type theory tended to get relegated to logic as a higher-order logic. Although this is an important and interesting topic, I will not consider it further here, as it would take me well beyond my primary concern with the philosophy of set theory.

[5] Many of the standard texts of this period clearly reflect this approach to set theory. See, for example, the many set theories studied in A. A. Fraenkel, Y. Bar-Hillel, and A. Levy's *Foundations of Set Theory*, 2nd rev. edn (Amsterdam: North-Holland, 1973), or Mendelson's *Introduction to Mathematical Logic* (Boca Raton, FL: CRC Press, 1997). As we will see, Quine's *Set Theory and Its Logic*, rev. edn (Cambridge, MA: Harvard University Press, 1969), is certainly part of this tradition in set theory. This is quite different from most contemporary set theory texts, which focus pretty strictly on Zermelo's set theory and its variations. This may now be starting to change again in light of the growing interest and value found in the study of non-well-founded set theories. Some standard introductory set theory texts are starting to cover this topic; see, for example, Keith Devlin, *The Joy of Sets: Fundamentals of Contemporary Set Theory*, 2nd edn (New York, NY: Springer-Verlag, 1993), Ch. 7, and Karel Hrbacek and Thomas Jech, *Introduction to Set Theory*, 3rd rev. and expanded edn (New York, NY: Marcel Dekker, 1999), Ch. 14, Sec. 3.

that had been set for a viable set theory. Such is the set theory that is the main topic of this book: Quine's *New Foundations*.

Quine first put forward his NF in the 1937 paper for which this set theory is named.[6] It is strikingly different from most other set theories in that it allows for a universal set, that is, a set of all sets.[7] Such "big" sets as this one, along with the set of ordinal numbers and the set of cardinal numbers, have often been thought to be the source of the set-theoretic paradoxes of Russell, Cesare Burali-Forti, and Cantor, respectively, and so have generally not received widespread attention in the literature. But as we will see in the opening chapters, such sets were not always so easily dismissed or altogether ignored. Early practitioners did consider such sets – not always favorably, but they were certainly not far from mainstream thinking about set theory, whereas they typically are thus positioned these days. Questions about these big sets were a real concern, raising questions that needed to be answered if set theory was to go on. For example, if big sets were the source of the paradoxes, on what grounds could they be ruled out as illegitimate? Or if they were to be included as part of the set-theoretic universe, how could they exist without giving rise to contradiction? These were genuine and important issues to consider during set theory's founding and development by Cantor, Russell, and Zermelo. In the more recent development of set theory, these issues were not so much resolved as just simply faded from view as the two main set theories – ZF set theory and Russell's theory of types – emerged, both denying the existence of these big sets.[8] With this, we lost any mathematical investigation into what Cantor identified as the absolute infinite.[9] As I have already noted, this early period of set theory was not one of dogmatism, and research in set theory remained fairly pluralistic through the 1950s. With many important questions still open, chief among them being the continuum hypothesis, no one axiomatization of set theory was taken as definitive, and much research

[6] W. V. Quine, "New Foundations for Mathematical Logic," *American Mathematical Monthly* 44:2 (1937), pp. 70–80.

[7] As we will see later, it is not quite as deviant as it is often portrayed to be. While it certainly does behave in unusual ways, particularly when its "big" sets are under consideration, it also develops fairly naturally out of insights gathered from Russell and Zermelo. We will see that, in a way, it is a generalization of ideas that Quine took from both of them.

[8] In his early work, Russell himself thought it near obvious that these big sets must exist, and as we will see, it was this belief that brought him to the discovery of Cantor's paradox.

[9] Cantor himself did not think that the absolute infinite could be investigated mathematically, but for theological rather than mathematical reasons.

concerned comparative accounts of the various axiomatizations that were available.[10] It was in this context of pluralism and tolerance that Quine introduced NF.

This book consists of seven chapters, divided into three main parts, though all related in some way. The first part of the book is concerned primarily with the early development of set theory, including the emergence of Quine's NF. In Chapter 1, I present a brief history of set theory from its beginnings with Cantor through to the discovery of the paradoxes, and then, in Chapter 2, I turn to the main proposals offered by Zermelo and Russell for resolving them. In these first two chapters, I aim to bring out two important points that I mentioned previously. The first is that to the extent to which there was any single shared conception of a set at the founding of the theory, it was the idea that a set is the extension of a predicate. The second point is that after the paradoxes, the success of a set theory was to be judged by its ability to explain the infinite and to serve as a framework for reconstructing ordinary non-set-theoretic mathematics. This gives rise to the approach to the philosophy of set theory that I describe as set theory as explication. This is the idea that there is no uniquely correct notion of a set beyond its being the sort of object that fulfills the minimal criteria that I have mentioned already: It is an extensional entity, and it must somehow be restricted enough to avoid the paradoxes but not so much so that it ceases to be capable of fulfilling its intended role in mathematics. I take this notion of an explication from Quine, who himself took it, as we will see, from Rudolf Carnap and Russell. Quine himself states in his book *Word and Object* that for an explication, "[w]e fix on the particular functions of the unclear expression that make it worth troubling about, and then devise a substitute, clear and couched in terms to our liking, that fills those functions."[11] In this view, I argue, set theory is developed pragmatically in an experimental and exploratory spirit. This approach is to be contrasted with what I describe as set theory as conceptual analysis. This view takes it as a given that there is a single correct set theory and, through conceptually analyzing the notion of set itself, we can discover what it is. This latter view appears to be dominant in the contemporary philosophy of set theory, with its adherence to the iterative conception of set and the general unwillingness to entertain set theories that run contrary to this

[10] Fraenkel and Bar-Hillel's *Foundations of Set Theory*, first published in 1958 with its last edition published in 1973, is a classic of this sort of work in set theory.
[11] W. V. Quine, *Word and Object* (Cambridge, MA: MIT Press, 1960), pp. 258–9.

conception.[12] We will see examples of this attitude mostly in later chapters, where I discuss NF and its accompanying philosophy in the context of the iterative conception.

In Chapter 3, which concludes the opening line of thought with its historical focus on the paradoxes, I present Quine's NF as combining the insights of both Zermelo and Russell into resolving the paradoxes. Here, I emphasize that Quine's exploratory and experimental approach fits squarely within the tradition of explication. Thus NF is far from an anomaly in the history of set theory, though it is often treated as such within the context of contemporary philosophy of set theory. In a way, then, the current dominance of the iterative conception is much more the anomaly in the history of set theory.

The second part of the book comprises two chapters in which I focus more directly on Quine's philosophy of set theory within the context of his philosophy as a whole. The focus here is not so much on NF specifically as on the philosophical thinking that informed Quine's approach to set theory generally. In Chapter 4, I focus primarily on how Quine's philosophy of set theory developed out his early engagement with Russell's logicism, particularly as put forward in Russell's *Principles of Mathematics* and the more technical working out of this project in Whitehead and Russell's *Principia Mathematica*.[13] Whereas Chapter 2 emphasizes Russell's foundational work as part of the tradition of set theory as explication, Chapter 4 tells a broader story, emphasizing some of the tensions between Russell's mathematical and philosophical approaches to the foundations of mathematics. It is in the former where we find the idea of set theory as an explication. I argue that we can understand Quine's own particular conception of set theory as explication as emerging out his attempt to come to terms with the tensions in Russell's work. The final part of Chapter 4 returns to NF to emphasize the philosophy that accompanies it. In particular, we see in Quine an

[12] There is nothing in the iterative conception that demands that it be approached from the side of conceptual analysis. After all, as I have already remarked, the iterative conception is now most commonly taken to be embodied in the axioms of Zermelo's theory, which originally emerged out of the explicative approach. Indeed, I take it that there are good mathematical reasons to now favor a set theory such as ZF, but these are not the reasons that are typically offered in favor of the iterative conception over other conceptions of set. More typical are appeals to its rather intuitive character. This is an argument that I reject, as we will see throughout this book in various ways.

[13] Bertrand Russell, *Principles of Mathematics*, 2nd edn (London: George Allen and Unwin Ltd., 1937[first edition 1903]). Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*. 3 vols. 2nd edn (Cambridge: Cambridge University Press, 1925, 1927 [first edition 1910, 1912, 1913]).

attitude that is very critical of what he identifies as the more metaphysical approaches to the philosophy of set theory.

In Chapter 5, I continue to focus on the philosophy that informs Quine's approach to set theory, but I turn to his more developed and more general approach to philosophy as a whole. The main theme here will be how Quine takes the simplification and clarification of our conceptual scheme as a main task for philosophy. His work in set theory is largely a contribution to this project, focusing on the mathematical portion of our conceptual scheme and reaching its culmination in *Set Theory and Its Logic*. Within the context of Chapter 5, I also discuss in some detail Quine's understanding of the notion of explication. Since my account of Quine often pushes him very close to Carnap, I conclude Chapter 5 by trying to locate where their differences lie, particularly with regard to their appeals to tolerance in their respective philosophies.

In Part III, I consider in more detail the other tradition in the philosophy of set theory, that of conceptual analysis, by examining in Chapter 6 some arguments in favor of the iterative conception of set, most notably championed by George Boolos. I argue that this preference ultimately rests on pragmatic considerations that support NF equally well. I then look at NF in more detail in this context to show that, like ZF-style set theories, it largely satisfies the criteria offered earlier for judging a set theory successful. That is, it appears to eliminate the paradoxes while also providing an account of the infinite and serving as a framework for reconstructing the rest of mathematics within it. In addition, I emphasize that contemporary set-theoretic research tends to focus on set theory itself as a subject for mathematical investigation and thus no longer limits itself to being merely a foundation for other branches of mathematics. NF, with its "big sets" such as the universal set, the set of cardinal numbers, and the set of ordinal numbers, is thus more consonant with current mathematical practice regarding set theory than might have been thought. It gives us a way to investigate the mathematical behavior of such sets, which ZF and related systems exclude. Again, my ultimate aim is not to argue for NF as the correct set theory over other set theories but to show that the concurrent development of various set theories furthers our understanding of the set-theoretic universe and so also our mathematical knowledge generally.

In Chapter 7, I examine some potential problems with NF. NF's development of arithmetic does not appear to proceed as elegantly as that of ZF. One way to deflect this challenge is to draw attention again to the fact that set theory has moved far beyond being merely a foundation for other branches of mathematics. I then also observe that one reason why ZF does give us an elegant account of, say, arithmetic is that ZF begins by ruling out certain troublesome sets from the start. In the more pragmatic view that I am putting forth, this a perfectly acceptable starting point. But I then argue that it would also be perfectly acceptable not to rule out these sets, particularly when our interest is in the mathematics of the set-theoretic universe as a whole. Within this context, I also consider the status of AC in NF. NF is notorious for disproving this axiom, which is used crucially throughout mathematics. However, I show that – much as in the case of arithmetic – this apparent failing arises only because NF includes sets that ZF does not. In NF, AC fails only in full generality. Appropriately restricted, enough of AC is present to satisfy the usual mathematical needs. Again, I show that such a failing of AC is hardly out of line with the set-theoretic tradition. For much of its history, AC has been treated carefully by set theorists, often flagging where it is used in proofs or in trying to re-prove results so that they do not rely on the axiom. Although much of contemporary set theory has come to treat AC as constitutive of the behavior of sets, even considering the principle obvious, this seems unfounded. After all, AC is well known for its highly unintuitive consequences, such as the Banach–Tarski paradox. In consequence, the truth of AC seems far from settled, thus leaving as a worthwhile endeavor the investigation of set theories for which it fails.

My aims in the book are broadly philosophical in that, most generally, I want to urge that logic and mathematics continue to be fruitful sources for thinking about philosophical problems that have been with us since philosophy's beginning. In particular, this book raises again the age-old question of how we come to acquire our nonempirical knowledge. In arguing that set theory is pragmatically developed, I want to suggest that mathematics in its foundations develops in ways that share much with the natural sciences and therefore is not necessarily the paradigm of a priori knowledge that philosophers have often thought it to be.

Before concluding this introduction, I should comment further on what I have been describing as an approach to set theory as explication. I have emphasized that this approach to set theory is pragmatic. Some of my exposition up to this point and in what follows may suggest that the pragmatic approach to set theory is unique to my account of set theory as explication and that it no longer has much hold on contemporary philosophy of set theory. This is not entirely true. A number of commentators have noted that this tradition has a firm place in the history of set theory.[14] Furthermore, a pragmatic approach to set theory is not entirely absent from contemporary set theory.[15] But in accord with the account that I have been laying out, this pragmatism happens only within ZF-style set theories. It does not extend to alternative axiomatizations. This is just the sort of dogmatism in set theory that I will question in what follows. With this in mind, let me now present Quine's NF and its place in the philosophy of set theory.

[14] For some examples of the pragmatic tradition in the history of set theory, see Gregory H. Moore, "The Origins of Zermelo's Axiomatization of Set Theory," *Journal of Philosophical Logic* 7:1 (1978), pp. 324–5; M. D. Potter, "Iterative Set Theory," *The Philosophical Quarterly* 43:171 (1993), p. 181; and Heinz-Dieter Ebbinhaus, *Ernst Zermelo: An Approach to His Life and Work* (New York, NY: Springer, 2007), p. 80. I should also add that I have previously described what I am now calling set theory as explication as the pragmatic approach to set theory. However, this label seemed to cause too much confusion with American pragmatism as a general philosophical position. For suggesting the new terminology, distinguishing between set theory as explication and set theory as conceptual analysis, I thank Gary Ebbs.

[15] We find this in Penelope Maddy's recent *Defending the Axioms: On the Philosophical Foundations of Set Theory* (New York, NY: Oxford University Press, 2011), in her discussions of intrinsic versus extrinsic justifications of the axioms of set theory. The former lines up with conceptual analysis and the latter with explication.