

## Probabilistic Mechanics of Quasibrittle Structures

### Strength, Lifetime, and Size Effect

Quasibrittle materials are becoming increasingly important for modern engineering. They include concretes, rocks, fiber composites, tough ceramics, sea ice, bone, wood, stiff soils, rigid foams, glass, dental and biomaterials, as well as all brittle materials on the micro or nano scale. Their salient feature is that the fracture process zone size is non-negligible compared to the structural dimensions. This causes intricate energetic and statistical size effects and leads to size-dependent probability distribution of strength, transitional between Gaussian and Weibullian.

*Probabilistic Mechanics of Quasibrittle Structures* discusses the ensuing difficult challenges for safe design.

Drawing upon years of practical experience and using numerous examples and illustrative applications, the authors cover:

- Rigorous theory with detailed derivations yet no superfluous mathematical sophistication.
- Extensive experimental verifications and realistic approximations for design.
- Fracture kinetics and its size effect.
- Multiscale analytical transition to the material scale.
- Statistics of structural strength and lifetime, size effect and reliability indices.
- Ramification to gate dielectrics breakdown with analogous mathematical formulation.

Born and educated in Prague (Ph.D. 1963), Zdeněk Bažant joined Northwestern University in 1969, where he has been W.P. Murphy Professor since 1990 and simultaneously McCormick Institute Professor since 2002, as well as Director of Center for Geomaterials (1981–1987). He was inducted to NAS, NAE, American Academy of Arts & Sciences, Royal Society London; to the academies of Italy, Austria, Spain, Czech Republic, Greece and Lombardy; and to Academia Europaea and European Academy of Sciences & Arts. He is an honorary member of ASCE, ASME, ACI, and RILEM; received seven honorary doctorates as well as the von Karman, Timoshenko, Prager and Newmark medals among many honors; was awarded the Austrian Cross for Science and Art 1st Class from president of Austria; and was president of SES, IA-FraMCoS and IA-ConCreep. He has authored six books and over 600 papers. In 2015, ASCE established ZP Bažant Medal for Failure and Damage Prevention, and ZP Bažant Prize for Engineering Mechanics was created in Czech Republic. He is one of the original top 100 ISI Highly Cited Scientists in Engineering ([www.ISIhighlycited.com](http://www.ISIhighlycited.com)).

Dr. Jia-Liang Le is currently Associate Professor of Civil, Environmental, and Geo-Engineering at the University of Minnesota. He obtained his Ph.D. in structural mechanics from Northwestern University in 2010. He received the Best Paper Award of the 48th U.S. Rock Mechanics/ Geomechanics Symposium, the 2015 Young Investigator Award from the U.S. Army Research Office, and the 2017 ASCE Leonardo da Vinci Award. His research interests include fracture mechanics, probabilistic mechanics, scaling, and structural reliability.

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**Dedicated to the memory of Alfred Martin Freudenthal (1906–1977), widely regarded as the founder of the field of structural safety and reliability, encompassing both the theories of probability and statistics and the theories of mechanics of materials and structures.**

# Contents

	<i>Foreword</i>	<i>page</i> xiii
	<i>Preface</i>	xv
<b>1</b>	<b>Introduction</b>	1
	1.1 The Problem of Tail of Probability Distribution	1
	1.2 History in Brief	3
	1.2.1 Classical History	3
	1.2.2 Recent Developments	6
	1.3 Safety Specifications in Concrete Design Codes and Embedded Obstacles to Probabilistic Analysis	9
	1.4 Importance of Size Effect for Strength Statistics	10
	1.5 Power-Law Scaling in the Absence of Characteristic Length	12
	1.5.1 Nominal Strength of Structure and Size Effect	13
	1.6 Statistical and Deterministic Size Effects	14
	1.7 Simple Models for Deterministic Size Effects	14
	1.7.1 Type 1 Size Effect for Failures at Crack Initiation	15
	1.7.2 Type 2 Size Effect for Structures with Deep Cracks or Notches	16
	1.8 Probability Distributions of Strength of Ductile and Brittle Structures	19
<b>2</b>	<b>Review of Classical Statistical Theory of Structural Strength and Structural Safety, and of Statistics Fundamentals</b>	22
	2.1 Weakest-Link Model	22
	2.2 Weibull Theory	23
	2.3 Scaling of Weibull Theory and Pure Statistical Size Effect	24
	2.4 Equivalent Number of Elements	26
	2.5 Stability Postulate of Extreme Value Statistics	27
	2.6 Distributions Ensuing from Stability Postulate	28
	2.7 Central Limit Theorem and Strength Distribution of Ductile Structures	30
	2.8 Failure Probability When Both the Strength and Load Are Random, and Freudenthal Integral	32

<b>3</b>	<b>Review of Fracture Mechanics and Deterministic Size Effect in Quasibrittle Structures</b>	<b>35</b>
3.1	Linear Elastic Fracture Mechanics	35
3.2	Cohesive Crack Model	37
3.3	Crack Band Model	40
3.4	Nonlocal Damage Models and Lattice-Particle Model	44
3.5	Overcoming Instability of Tests of Post-Peak Softening of Fiber-Polymer Composites	46
3.6	Dimensional Analysis of Asymptotic Size Effects	47
3.7	Second-Order Asymptotic Properties of Cohesive Crack or Crack Band Models	50
3.8	Types of Size Effect Distinguished by Asymptotic Properties	51
3.9	Derivation of Quasibrittle Deterministic Size Effect from Equivalent LEFM	52
3.9.1	Type 2 Size Effect	53
3.9.2	Type 1 Size Effect	54
3.10	Nonlocal Weibull Theory for Mean Response	56
3.11	Combined Energetic-Statistical Size Effect Law and Bridging of Type 1 and 2 Size Effects	57
<b>4</b>	<b>Failure Statistics of Nanoscale Structures</b>	<b>59</b>
4.1	Background of Modeling of Nanoscale Fracture	59
4.2	Stress-Driven Fracture of Nanoscale Structures	60
4.3	Probability Distribution of Fatigue Strength at Nanoscale	65
4.4	Random Walk Aspect of Failure of Nanoscale Structures	66
<b>5</b>	<b>Nano-Macroscale Bridging of Probability Distributions of Static and Fatigue Strengths</b>	<b>71</b>
5.1	Chain Model	72
5.2	Fiber-Bundle Model for Static Strength	73
5.2.1	Brittle Bundle	74
5.2.2	Plastic Bundle	79
5.2.3	Softening Bundle with Linear Softening Behavior	81
5.2.4	Bundle with General Softening Behavior and Nonlocal Interaction	84
5.3	Fiber-Bundle Model for Fatigue Strength	88
5.4	Hierarchical Model for Static Strength	92
5.5	Hierarchical Model for Fatigue Strength	97
<b>6</b>	<b>Multiscale Modeling of Fracture Kinetics and Size Effect under Static and Cyclic Fatigue</b>	<b>100</b>
6.1	Previous Studies of Fracture Kinetics	100
6.2	Fracture Kinetics at Nanoscale	102

	6.3 Multiscale Transition of Fracture Kinetics for Static Fatigue	103
	6.4 Size Effect on Fracture Kinetics under Static Fatigue	106
	6.5 Multiscale Transition of Fracture Kinetics under Cyclic Fatigue	108
	6.6 Size Effect on Fatigue Crack Growth Rate and Experimental Evidence	112
	6.7 Microplane Model for Size Effect on Fatigue Kinetics under General Loading	117
<b>7</b>	<b>Size Effect on Probability Distributions of Strength and Lifetime of Quasibrittle Structures</b>	<b>119</b>
	7.1 Probability Distribution of Structural Strength	119
	7.2 Probability Distribution of Structural Lifetime	122
	7.2.1 Creep Lifetime	122
	7.2.2 Fatigue Lifetime	127
	7.3 Size Effect on Mean Structural Strength	129
	7.4 Size Effects on Mean Structural Lifetimes and Stress-Life Curves	133
	7.5 Effect of Temperature on Strength and Lifetime Distributions	136
<b>8</b>	<b>Computation of Probability Distributions of Structural Strength and Lifetime</b>	<b>139</b>
	8.1 Nonlocal Boundary Layer Model for Strength and Lifetime Distributions	139
	8.2 Computation by Pseudo-random Placing of RVEs	144
	8.3 Approximate Closed-Form Expression for Strength and Lifetime Distributions	146
	8.4 Analysis of Strength Statistics of Beams under Flexural Loading	152
	8.5 Optimum Fits of Strength and Lifetime Histograms	154
	8.5.1 Optimum Fits of Strength Histograms	154
	8.5.2 Optimum Fits of Histograms of Creep Lifetime	157
	8.5.3 Optimum Fits of Histograms of Fatigue Lifetime	159
<b>9</b>	<b>Indirect Determination of Strength Statistics of Quasibrittle Structures</b>	<b>161</b>
	9.1 Relation between Mean Size Effect Curve and Probability Distribution of RVE Strength	161
	9.2 Experimental Verification	164
	9.2.1 Description of Experiments	164
	9.2.2 Analysis of Test Results	166
	9.3 Determination of Large-Size Asymptotic Properties of the Size Effect Curve	169
	9.4 Comparison with the Histogram Testing Method	170
	9.5 Problems with the Three-Parameter Weibull Distribution of Strength	171
	9.5.1 Theoretical Argument	171
	9.5.2 Evidence from Histogram Testing	172
	9.5.3 Mean Size Effect Analysis	173

x	<b>Contents</b>	
	9.6 Alternative Proof of Strength Distribution of an RVE Based on Stability Postulate and Atomistic Analysis	176
<b>10</b>	<b>Statistical Distribution and Size Effect on Residual Strength after Sustained Load</b>	<b>177</b>
	10.1 Nanomechanics Based Relation between Monotonic Strength and Residual Strength of One RVE	178
	10.2 Analysis of Residual Strength Degradation for One RVE	180
	10.3 Probability Distribution of Residual Strength	181
	10.3.1 Formulation of Statistics of Residual Strength for One RVE	181
	10.3.2 Formulation of Residual Strength cdf of Geometrically Similar Structures of Different Sizes	182
	10.4 Comparison among Strength, Residual Strength, and Lifetime Distributions	183
	10.5 Experimental Validation	184
	10.5.1 Optimum Fits of Strength and Residual Strength Histograms of Borosilicate Glass	184
	10.5.2 Optimum Fits of Strength Histograms and Prediction of Lifetime and Mean Residual Strength for Unidirectional Glass/Epoxy Composites	186
	10.5.3 Prediction of Strength Degradation Curve for Soda-Lime Silicate Glasses	188
	10.6 Comparison of Size Effects on Mean Strength, Residual Strength, and Lifetime	189
<b>11</b>	<b>Size Effect on Reliability Indices and Safety Factors</b>	<b>193</b>
	11.1 Size Effect on the Cornell Reliability Index	194
	11.2 Size Effect on the Hasofer–Lind Reliability Index	197
	11.3 Approximate Equation for Scaling of Safety Factors	199
	11.4 Analysis of Failure Statistics of the Malpasset Arch Dam	202
	11.4.1 Model Description	203
	11.4.2 Discussion of Cornell and Hasofer–Lind Indices	204
	11.4.3 Discussion of Central and Nominal Safety Factors	207
<b>12</b>	<b>Crack Length Effect on Scaling of Structural Strength and Type 1 to 2 Transition</b>	<b>210</b>
	12.1 Type 1 Size Effect in Terms of Boundary Strain Gradient	211
	12.2 Universal Size Effect Law	213
	12.3 Verification of the Universal Size Effect Law by Comprehensive Fracture Tests	215
<b>13</b>	<b>Effect of Stress Singularities on Scaling of Structural Strength</b>	<b>218</b>
	13.1 Strength Scaling of Structures with a V-Notch under Mode I Loading	218



	13.1.1 Energetic Scaling of Strength of Structures with Strong Stress Singularities	219
	13.1.2 Generalized Finite Weakest-Link Model	220
	13.2 Numerical Simulation of Mode I Fracture of Beams with a V-Notch	223
	13.2.1 Model Description	223
	13.2.2 Results and Discussion	224
	13.3 Scaling of Fracture of Bimaterial Hybrid Structures	228
	13.3.1 Energetic Scaling with Superposed Multiple Stress Singularities	229
	13.3.2 Finite Weakest-Link Model for Failure of Bimaterial Interface	232
	13.4 Numerical Analysis of Bimaterial Fracture	234
	13.4.1 Description of Analysis	234
	13.4.2 Results and Discussion	236
<b>14</b>	<b>Lifetime of High-<math>k</math> Gate Dielectrics and Analogy with Failure Statistics of Quasibrittle Structures</b>	<b>239</b>
	14.1 Deviation of Lifetime Histograms of High- $k$ Dielectrics from the Weibull Distribution	239
	14.2 Breakdown Probability	242
	14.2.1 Analogy with Strength of Quasibrittle Structures	242
	14.2.2 Application to Dielectric Breakdown	244
	14.2.3 Microscopic Statistical Models	245
	14.2.4 Breakdown Voltage Distribution	248
	14.3 Breakdown Lifetime under Constant Voltage	249
	14.3.1 Relation between Lifetime and Breakdown Voltage	249
	14.3.2 Microscopic Physics	250
	14.3.3 Probability Distribution of Breakdown Lifetime	251
	14.4 Breakdown Lifetime under Unipolar AC Voltage	251
	14.5 Experimental Validation	252
	14.5.1 Breakdown under Constant Gate Voltage Stress	252
	14.5.2 Breakdown under Unipolar AC Voltage Stress	255
	14.6 Size Effect on Mean Breakdown Lifetime	255
	<b>Appendix A: Power-Law Scaling of Boundary Value Problems</b>	<b>257</b>
	<b>Appendix B: Proof of Transitional Size Effects of Types 1 and 2 by Dimensional Analysis and Asymptotic Matching up to Second Order</b>	<b>260</b>
	<b>Appendix C: Proof of Small-Size Asymptotics of Cohesive Crack Model up to Second Order</b>	<b>264</b>
	<i>References</i>	269
	<i>Author Index</i>	291
	<i>Subject Index</i>	297

## Foreword

Failure of materials was recognized centuries ago as a critical component of structural mechanics and consequently of structural design. Efficient (minimum amount of materials) and economical (minimum cost) designs are based on the basic principle of taking full advantage of the strength of the materials used, while at the same time carefully avoiding any type of material failure.

Whether or not they were able to quantify it, adequately, engineers realized very soon that material failure involved a high level of uncertainty. Later on, based on advances in the mathematical theory of probability, researchers in the field of mechanics identified two basic types of material behavior as far as failure is concerned – ductile and brittle failures – and were able to establish rigorous probabilistic models for both. The Gaussian and Weibull probability distributions have been the standard models for these two types of material failure, respectively.

However, there has always been a transition area between ductile and brittle failures with a behavior that was orders of magnitude more complex and challenging to model: it has been named “quasibrittle behavior.” The two standard models for ductile and brittle failure were clearly not adequate in this transition area, and on top of that, there is a very wide range of materials falling in this category at the scale of laboratory testing and normal structures: concrete, various composites, toughened ceramics, many rocks, coal, ice, rigid foams, biological shells, bone, cartilage, dental ceramics, and many others. Furthermore, at the nano- and micrometer scales, virtually all materials become quasibrittle.

The first author of this book – one of the giants in the field of mechanics – has been instrumental over the years in developing a rigorous theoretical framework modeling the failures of structures made of quasibrittle materials. In a long series of seminal scientific papers, he has identified the challenges involved and has introduced a number of groundbreaking theories and models to address them. He and his coauthor, who, despite being much younger, has already impacted the field by his own seminal contributions, have provided in this volume the definitive treatment of this formidably challenging field, and in the process have established the complete theory of any type of material failure, ranging between the two limiting cases of ductile and brittle behaviors.

In particular, I would like to highlight the equal emphasis and importance given by the authors to the two disciplines of mechanics and probability, and their ingenious and highly successful blending of the two in a fully integrated theoretical framework. The

volume is a true pleasure to read and will become immediately an indispensable tool for every scientist, scholar, and engineer interested in this critically important field.

George Deodatis  
Santiago and Robertina Calatrava Family Endowed Chair and Chair,  
Department of Civil Engineering and Engineering Mechanics, Columbia University  
President, International Association for Structural Safety and Reliability, 2009–2013

## Preface

Although some would vehemently deny it, many specialists would agree that, since the 1977 death of Freudenthal,<sup>1</sup> the research field of structural safety and reliability has been in a schism.

Alfred Freudenthal, the founder of this field of research in the 1960s, perceived the fields of (1) structural safety and (2) mechanics and physics of materials and structures as inseparable. He mastered both, and treated both to the depth of knowledge in his time. Since that time, unfortunately, most researchers have immersed themselves in one of these two fields in great detail and with high sophistication, while treating the other aspect simplistically and superficially. The connection has been weak.

On one side, there have been probabilists who develop and successfully market complex computer programs to assess safety, reliability, and lifetime of concrete structures without recognizing that failure probability of concrete structures cannot be predicted with simplistic or obsolete material models that eschew fracture mechanics and energetic size effect. Or there have been statistically minded experimenters who conduct extensive histogram testing of the strength of ceramics but ignore the scale effects, micromechanics, and microscale physics of failure.

On the other side, there have been mechanicians who construct highly refined constitutive and computational models for the mechanics of failure of concrete, geomaterials,

<sup>1</sup> Born in Poland in 1906, Alfred Martin Freudenthal received his engineering and doctoral degrees in Prague, in 1929 and 1930, respectively. His dissertation dealt with the theory of plasticity. He worked in Prague as a structural engineer in a well-known engineering design firm, collaborated with Prof. J. Melan, a leading bridge designer at that time, and simultaneously, in 1934, collected another engineering degree in Lwow. From 1936 to 1946 he was one of the chief engineers of the new port of Tel Aviv and, after 1938, also a professor of bridge engineering at the Hebrew University of Technology in Haifa. In 1948 he arrived in the United States as a visiting professor at the University of Illinois, and between 1949 and 1969 he was a professor of civil engineering at Columbia University, New York, and then, until his death in 1977, a professor at George Washington University, Washington, DC.

A selection of his papers, published in 1981 by the American Society of Civil Engineers, shows a remarkably evenhanded attention to the mechanical behavior of materials and structures on one side, and the structural safety, reliability and fatigue on the other side, with both aspects intertwined as if they were one. His seminal works deal with viscoelasticity and nonlinear creep, plastic shells, orthotropic sandwich plates and shells, shrinkage stresses, concrete creep, consolidating media, strength of airframes, shear dilatancy in rock, seismic waves, work-hardening law for metals, relaxation spectra, second-order strain effects in metals, physical and statistical aspects of metal fatigue and residual stresses, fundamental theory of structural safety, safety of prestressed concrete, cumulative damage, lifetime estimation, random failure of structures with multiple load paths, reliability of reactor components, extreme value risk analysis, reliability of aircraft and of offshore platforms in seismic regions, structural optimization, and risk control.

and composites without recognizing that far greater prediction errors stem from simplistic or nonexistent treatment of the randomness of the material as well as the loads.

The present book attempts a step to rectify this schism. In a unified theoretical framework, it deals with the quasibrittle structures, which are those consisting of quasibrittle (or brittle heterogeneous) materials. These are commonplace materials, used more and more widely and increasingly important for modern technology, including much of high-tech. They encompass concretes (as the archetypical case), rocks, fiber composites, tough ceramics, sea ice, bone, wood, stiff soils, rigid foams, and so forth, as well as all brittle materials on the micrometer scale. They are characterized by a fracture process zone that is not negligible compared to the typical structural dimensions. This feature causes an intricate energetic size effect, which is intertwined with the classical statistical size effect, the only kind of size effect known in classical fracture mechanics of brittle materials. Compared to metals and ceramics, a probabilistic theory of strength, lifetime, and size effect of quasibrittle structures has been developed much more recently. Its comprehensive presentation, based mostly on previous studies at Northwestern University and the University of Minnesota, is the objective of this book.

Although the main purpose of this book is a comprehensive mathematical exposition of the subject, the book is also suitable as a text for an advanced course, as all the results are mathematically derived and the focus is on understanding rather than just description. Moreover, parts of the book can be covered in graduate courses dealing with the modeling of failure of various materials and structures, as featured in the curricula in civil, mechanical, aerospace, nuclear, offshore, geotechnical, and ocean engineering, as well as materials science and geophysics. Chapters 1–3, Sections 6.1, 6.4, 6.6, 12.1, 12.2, 13.1, 13.3, and Appendices A–C can also be covered in a broader graduate course on Quasibrittle (or Cohesive) Fracture and Scaling, which the first author has been teaching at Northwestern since the 1980s, and which the second author later introduced at the University of Minnesota.

Working on the journal articles that underlie our book, we benefited from outstanding collaborators, particularly Sze-Dai Pang, Marco Salviato, Miroslav Vořechovský, Jan Eliáš, Drahoš Novák, Augusto Cannoe Falchetto, Mihai Marasteanu, Joseph Labuz, Roberto Ballarini, Johnathan Manning, Bing Xue, and Mathieu Pieuchot. They deserve our deep thanks. The results presented here could not have been achieved without generous funding from the US National Science Foundation, the US Department of Energy, the Army Research Office, the US Department of Transportation, the Boeing Co., the Minnesota Department of Transportation, and the Center for Transportation Studies at the University of Minnesota, for all of which we are very grateful.

We also want to express our deep appreciation of the stimulating research environments provided by Northwestern University and the University of Minnesota. Last but not least, we wish to express our wholehearted thanks to our wives, Iva M. Bažant and Miao Pan, for their loving support of our research endeavor.

Zdeněk P. Bažant and Jia-Liang Le  
Evanston and Minneapolis, July 18, 2016