Part I

Overview

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Excerpt
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Why Analytical Mechanics?

Analytical mechanics is a branch of dynamics that is concerned with describing moving things in terms of analytical formulas rather than geometrical diagrams. It is a mathematical extension of the fundamental ideas of Newtonian mechanics that is extremely useful for formulating, solving, and interpreting the motions of mechanical systems. For a proper beginning, therefore, it is useful to revisit the physical context that we are mathematically extending. I will assume you have had a pretty good grounding in basic physics, although, strictly speaking, I don't even know who you are. The current chapter is meant to review, rather than introduce, some of the basics.

1.1 Broad Concepts

The great achievement of Newton (and the giants on whose shoulders he left his footprints) was to articulate the fundamental physical concept that governs the behavior of mechanical systems. Abstracted from phenomena and reduced to its perfect, idealized, Platonic archetype, this idea addresses an idealized point particle, that is, a mass sufficiently small that you don't need to know how big it is or what shape it has. Its motion can adequately be described by a single, time-dependent coordinate $\mathbf{r}(t)$; you don't need two coordinates to keep track of both its ends, for example.[1]

Suppose this pint-size particle has mass m and is acted on by a force $\mathbf{F}$. The physical content of Newton's second law is that m and $\mathbf{F}$ taken together determine the way in which the particle's velocity changes. That is, under the influence of this force, the mass's velocity $\mathbf{v} = \dot{\mathbf{r}}$ is about to change, according to

[1] Here and throughout this book, boldface letters stand for vectors in three-dimensional space.

$$m\frac{d\dot{\mathbf{r}}}{dt} = \mathbf{F} \quad \text{or}$$
$$m\ddot{\mathbf{r}} = \mathbf{F}. \tag{1.1}$$

Here we use the convention that each dot over a symbol denotes a time derivative of that symbol. Equation (1.1) is a big deal and not immediately obvious: it *could have been* that the force is proportional to the velocity $\dot{\mathbf{r}}$ itself, or to, say, the third derivative $\dddot{\mathbf{r}}$. But it just isn't; careful observation and measurement of masses moving subject to various everyday forces convinces us that (1.1) is a correct description of how things really move. Such is the physical content of what is colloquially known as "$F = ma$." Having established this, Newtonian mechanics goes over to mathematics and to the solution of the differential equations implied by (1.1). Note that this is actually three differential equations, for three components of the vector $\mathbf{r}$.

And in a way that's all you need. If you happen to consider the mechanics of something that is not a point particle, well, to a very good approximation you can consider it to be *made up of* a whole lot of little point particles. For example, you might want to calculate the motion of a bridge swaying in the wind. A bridge is definitely not a point particle, especially at rush hour. So, you conceptually divide the mass up into a finite collection of little masses that fill up the volume of the bridge, and follow the motion of all of them. Each such little mass m_i, at location $\mathbf{r}_i$, still satisfies Newton's equation

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i,$$

bearing in mind that this force $\mathbf{F}_i$ is the net force acting on mass m_i. That is, $\mathbf{F}_i$ is the sum of *all* the forces on m_i, including the forces of the wind, and gravity, and all the forces of all the other masses m_j that push and pull on m_i. This is definitely a complication, since now we have to solve a large number of differential equations, three for each mass, and we have to solve them all at the same time, since the forces between the masses probably depend on where they are relative to one another. Still, this is not really a big deal; computers handle this kind of thing all the time. Our point here is merely that Newton's basic equation is the only fundamental concept you need to establish the equations of motion for any mechanical system you can dream up (neglecting, as we will in most of this book, relativistic systems).

So, why would you need anything else? Why go on and invent analytical mechanics, which, as we will see in the following pages, involves its own irritating complications? There are several reasons, to be developed over the course of the book, but for now let's give a quick overview. We will not belabor these points here, as we will have adequate opportunity to belabor them later.

1.1.1 Forces of Constraint

A pendulum, as you know, is a mass tied to the ceiling by a string and that can swing back and forth (Figure 1.1). We will have a lot more to say about pendulums in Chapter 2, but for now just consider applying Newton's law to this mass. To do so, you would have to know the net force on the mass. One of the forces acting is clearly the weight of the mass, which is known and can be specified ahead of time. It is one of the things, along with the length of the string, that *determines* the problem you are solving.

The other force is the tension force that the string exerts on the mass. This is a force of constraint: the string constrains the mass to move on a circle whose radius is the length of the string. What is this force? The answer is, you don't know this ahead of time, and indeed this force will turn out to change over time as the mass moves. It is one of the things *determined by* the problem you are solving. So in the end, solving $F = ma$ directly, you would have to work out four equations for four unknowns – two coordinates, plus two components of the a priori unknown force of tension.

This kind of thing quickly adds up, particularly if you think about more complicated things than pendulums, where lots of masses might have lots of constraints (the little pieces of the bridge, for example). One of the genius aspects of analytical mechanics is to eliminate the need to deal with these forces of constraint. It's not merely that you can deal with them more easily;

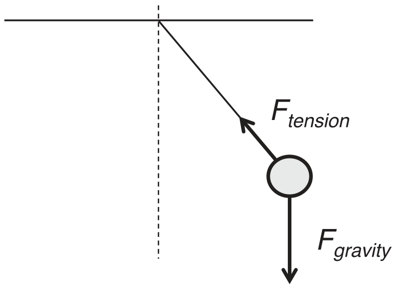

Figure 1.1 Forces on a pendulum.

it's really that you can solve problems without ever considering the forces of constraint at all. You can go back afterwards and get them if you want to, but let's face it, you hardly ever want to.

1.1.2 Collective Coordinates

Here's another example. It is also common practice to model the behavior of liquids by tracking the motion of the molecules that make up the liquid, assumed to interact by complicated intermolecular forces that are somehow known. These calculations can be done for hundreds of thousands of molecules at a time. The result is complete and detailed knowledge of hundreds of thousands of coordinates of molecules as they evolve in time. What are we to make of this? All these coordinates are much too much information to take in all at once. In practice, one uses these trajectories to compute simpler, collective observables, like the pressure of the gas, or some kind of correlation function that measures how likely you are to find two particles at a given distance from one another. Therefore, even though the calculation is complete and accurate, you are interested in the end in simpler, gross features.

In a way, analytical mechanics is a means of simplifying complicated calculations while retaining accuracy. A main procedure in analytical mechanics is to reduce the full description of all the coordinates of all the particles to a simplified description of only those coordinates that really matter. What "matters" of course follows from what the problem is and what you expect to get out of it.

An example with which you are probably familiar is the removal of center of mass coordinates for a collection of moving particles. Let's say a star and a very large planet orbit each other. A possible trajectory for this motion is shown in Figure 1.2. These curves show the trajectories of the star and the planet as viewed from some fixed position by an observer watching them move by. To draw this figure we require four coordinates for the two celestial objects, in the plane of their motion. It is not necessarily clear from this figure that they are going around each other. It's more like they are jerking each other back and forth.

This is a perfectly reasonable and useful way to describe this motion. Suppose you could not see the planet using a telescope far away on Earth. You could nevertheless notice the wobbling motion of the visible star. From this you could infer the existence of the invisible planet, along with its mass and details of its orbit. This is indeed exactly how the first exoplanets (big ones!) were discovered in 1995.

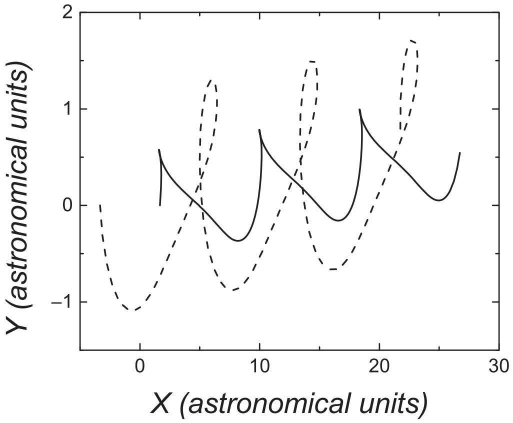

Figure 1.2 Simulated trajectories of a large planet (dashed line) and the star that it orbits (solid line), as seen from some point past which this miniature stellar system is moving.

Nevertheless, the mathematical description of bodies in orbit is usually much easier in a different, well-chosen set of coordinates. If the star and the planet have masses m_1 and m_2, respectively, and are located at coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$, then you define new coordinates

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$
$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1. \tag{1.2}$$

At this point, we will not worry about the form of the equations $F = ma$ as written in these coordinates, nor about their solution, but will jump directly to the result. The trajectories in these coordinates are pretty simple, as shown in Figure 1.3. The center of mass coordinate **R** just moves off at constant velocity in a straight line, while the relative coordinate **r** shows in a very explicit way the elliptical orbit that you just knew was hiding in there somewhere. The coordinates that really help understand orbits are the three components of **r**; we do not really have to fret over the three auxiliary coordinates of **R**, unless later on we want to draw figures like Figure 1.2.

This extremely useful set of coordinates displays a fairly odd yet general feature: the description of this particular mechanical system is simplified considerably by identifying coordinates that are *not the coordinates of any of the individual masses*. This idea is emblematic of the value of analytical mechanics. In the simplest view (which is the one we adopt in this book), the job of analytical mechanics is to provide the mathematical tools to exploit

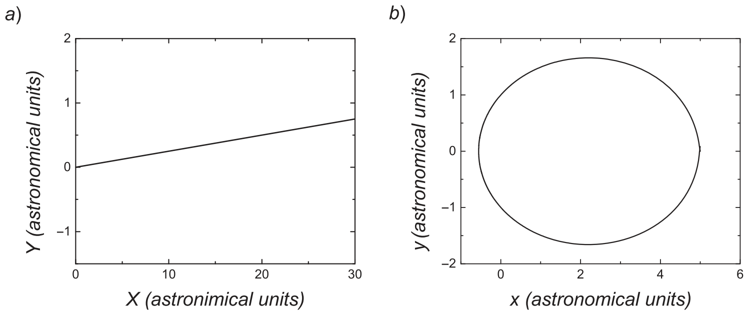

Figure 1.3 These curves represent mathematically the same motion of the star and planet as in Figure 1.2, but in very different coordinates. Here, the center of mass coordinate moves in a straight line, whereas the relative coordinate traces out one of our favorite conic sections, an ellipse.

strange new coordinate systems that you might find useful in describing your problem.

1.1.3 Momentum

While we're at it, these orbiting celestial objects would be a great opportunity to introduce another of the central concepts of analytical mechanics, momentum. The momentum of a point particle is defined as

$$\mathbf{p} = m\dot{\mathbf{r}},$$

which doesn't seem to advance anything beyond considering the velocity itself. However, momentum becomes a distinct and interesting concept when there is more than one mass involved.

To see this, let's suppose that this planet and star travel through an otherwise empty area of the galaxy, like the Gamma Quadrant, where forces due to other stars are negligible. That is, the only forces acting on these objects are the gravitational forces they exert on each other. Then Newton's equations for the two objects are[2]

$$m_1 \frac{d\mathbf{v}_1}{dt} = \mathbf{F}_{1,2}$$
$$m_2 \frac{d\mathbf{v}_2}{dt} = \mathbf{F}_{2,1}$$

[2] Notice that here we are treating these gigantic things as point particles, because in this context we only care about where they are going, not any details like the rotation of the star, for example.

where in this notation $\mathbf{F}_{1,2}$ describes the force *on* mass 1 *due to* mass 2, and vice versa. But wait! By Newton's third law, these forces are equal and opposite, $\mathbf{F}_{1,2} = -\mathbf{F}_{2,1}$ – a general feature of most of the forces we deal with, certainly of gravity. In this case, the sum of the equations of motion is independent of the forces:

$$m_1 \frac{d\mathbf{v}_1}{dt} + m_2 \frac{d\mathbf{v}_2}{dt} = 0,$$

or

$$\frac{d}{dt}\left(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \right) = 0.$$

This in turn means that the total momentum of both objects together, $\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ is a conserved quantity – that is, it does not depend on time.

Conserved quantities are great – always find and exploit them if you can. In this case, we can make a similar argument as was advanced above. If we should happen to know the total momentum $\mathbf{P}$ (which we only need to do once, because it never changes), then observing the motion of the star, $\mathbf{v}_1$, is enough to automatically determine the motion of the planet $\mathbf{v}_2$, if both masses are known. It's like getting something for nothing!

By contrast, velocity is *not*, in general, conserved in this case. Noting that $\mathbf{v}_2 = \mathbf{P}/m_2 - (m_1/m_2)\mathbf{v}_1$, and that $\dot{\mathbf{P}} = 0$, the rate of change of the total velocity is

$$\frac{d}{dt}\left(\mathbf{v}_1 + \mathbf{v}_2 \right) = \left(1 - \frac{m_1}{m_2} \right) \dot{\mathbf{v}}_1,$$

and this is not necessarily equal to zero. If the masses are the same, then sure, total momentum and total velocity are still proportional and there's no big difference. Likewise, if neither mass accelerates, then the total velocity is constant. But in general, total velocity can change in some crazy way in time, whereas the total momentum is constant. Moreover, the conservation of total momentum here is tied to a physically reasonable and understandable circumstance, namely that the external force that would change the total momentum is zero.

In further developments of analytical mechanics, one seeks a momentum, or some kind of generalization of momentum, like angular momentum, for instance. It is often a useful idea to hang your thinking on, and also is often a conserved quantity for some physically sound reason. Once we get to Lagrangians, in fact, it will emerge in a pretty transparent way how to define momenta so that you can see, automatically in the formulas, whether the momentum you're looking at is conserved or not.

1.1.4 Energy

Not all physics problems actually require you to solve Newton's equations in their entirety. A lot of times the way you approach the problem depends on the question you're asking. This can be seen in a very simple example: suppose you are standing on a balcony, throwing a water balloon straight down on your friend, as in Figure 1.4. How does it get from your hand to the ground? (For simplicity, we ignore your friend's height and use the ground as our intended target. Maybe it's his shoes you are after.)

Let's measure the height of the balloon as a coordinate x from the ground, using the coordinate system in the figure. Then $F = ma$ for this mass is

$$m\ddot{x} = -mg, \tag{1.3}$$

whose solution is something you should have seen in your previous studies of physics. It is

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + h.$$

This is solved in a convention where the mass starts at height $x(t = 0) = h$, and is thrown with velocity v_0, where v_0 is positive when you throw the ball upward and negative when you throw the ball downward. This is the complete solution, the legacy of Newton in this case, and from it you can derive anything else you need to know about the motion of the water balloon. The dependence of the trajectory on the initial height h and velocity v_0 is made explicit, so if you want, you can try various things, like see the effect of throwing harder.

However, if you pose a different question, you can answer it in a seemingly different way. For example, suppose the thing you care most about is how

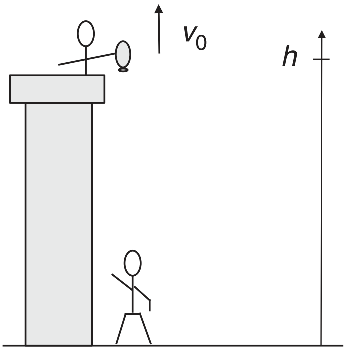

Figure 1.4 Look out below! Tracking a water balloon to the ground.