PART I

FUNDAMENTALS OF PASSIVE SEISMIC MONITORING

There is a crack in everything, that's how the light gets in.

Leonard Cohen (Anthem, 1992)

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Excerpt
More Information

1 Constitutive Relations and Elastic Deformation

> In the beginning, God said let the four-dimensional divergence of an antisymmetric second-rank tensor equal zero . . . and there was light.
>
> Michio Kaku (The Universe in a Nutshell, 2012)

Constitutive relations provide a foundation upon which to construct a theoretical framework for the response of a system to external stimuli. Formally, a constitutive relation defines the mathematical relationship between physical quantities that determine the response of a given material to applied forces (Macosko, 1994). In general, constitutive relations are based on experimental observation or mathematical reasoning other than a fundamental conservation equation (Pinder and Gray, 2008). This chapter deals primarily with a particular constitutive relationship that applies to *elastic media*; this relationship, known as the *generalized Hooke's Law*, describes a linear deformation regime in which the response to applied forces is fully recoverable and proportional to the magnitude of the net force. Countless experimental results confirm the applicability of this relationship to Earth materials when subject to small strains. As outlined in subsequent chapters, combining this constitutive relation with a few basic physical principles and boundary conditions leads to a remarkable wealth of wave-propagation phenomena.

As well as a description of the constitutive relations for an anisotropic elastic continuum, this chapter provides a brief introduction to various effective-medium theories that can be used to represent a complex medium with models that are more easily described and characterized. The types of media considered are of particular interest for investigations of reservoir processes and induced seismicity in sedimentary basins, including multiphase materials, vertically inhomogeneous (stratified) media and fractured elastic media. In addition, constitutive relationships for a poroelastic medium are introduced. This type of medium has two components: an elastic frame, plus a network of fluid-filled pores. A mathematical framework is also briefly introduced that governs the diffusion of pore-pressure in a poroelastic medium.

1.1 Stress and Strain

Forces that operate in Earth's interior drive a variety of deformation processes. The net internal force per unit area that acts at a point $\mathbf{x}$ on an arbitrary surface within a medium is called the *traction*, denoted by the vector $\mathbf{T}(\mathbf{x})$ (Figure 1.1). The surface on which this is defined may not necessarily correspond with a boundary, like a fracture or bedding plane.

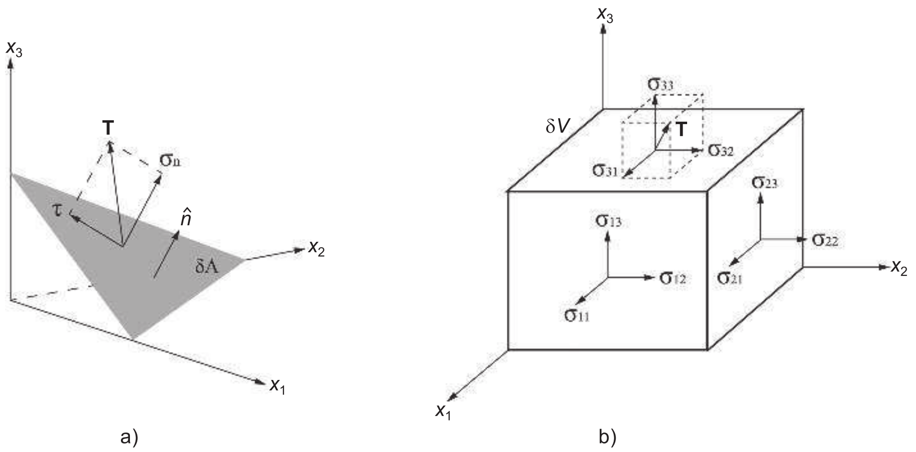

Fig. 1.1 Elements of the stress tensor. a) The traction acting on the shaded surface, denoted by **T**, can be decomposed into shear and normal components, denoted by τ and σ_n, respectively. b) Components of the stress tensor. δV represents an elementary volume.

To avoid the need to specify information about surface orientation, it is convenient to represent internal forces more generally using the *stress tensor*, which can be expressed with respect to an elementary volume as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} . \tag{1.1}$$

For each element of the stress tensor, the first index denotes the direction of the axis that is normal to the respective face for the elementary volume, while the second index denotes the direction in which the stress component acts. Both stress and traction have SI units of Pascal (Pa = N/m^2). For a surface with unit normal vector $\hat{\mathbf{n}}$, the stress tensor and jth element of the traction vector are related by

$$T_j = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \sum_{i=1}^{3} \sigma_{ij}\hat{n}_i \equiv \sigma_{ij}\hat{n}_i \, , \tag{1.2}$$

where the standard tensor summation convention for repeated indices is employed in the expression on the right side of Equation 1.2 (see Box 1.1). It follows that the net force **F** acting on a volume within a closed surface S may be written as

$$F_j = \int_S T_j dS = \int_S \sigma_{ij}\hat{n}_i dS \, . \tag{1.3}$$

In a state of equilibrium, force balance applies to the volume enclosed by S such that $|\mathbf{F}| = 0$.

The stress tensor has a number of salient characteristics. Maintaining continuity of a medium implies a condition of zero net torque on an elementary volume. This condition, in turn, implies that the stress tensor is symmetric (i.e. $\sigma_{ij} = \sigma_{ji}$). For a given surface

Box 1.1 Tensor Notation

Tensors are a generalization of vectors and provide a multidimensional representation of physical quantities that depend on spatial coordinates, including direction. Tensors are widely used in geophysics, as they are important for describing the physical properties of fields and systems in the disciplines of continuum mechanics and fluid mechanics. Tensor components are represented using index notation, where the *order* (also called *rank*) denotes the number of required indices. The stress tensor σ is a second-order tensor, sometimes referred to as a *dyadic*, and its ijth component is written as σ_{ij}. The *summation convention* for repeated indices, also known as *Einstein summation convention*, is used throughout this book. In this shorthand notation, repeated indices within products of tensors imply summation. Thus, the use of the summation convention means that

$$a_{ij}b_{jk} \equiv \sum_{j=1}^{N} a_{ij}b_{jk},$$

where N is the number of dimensions in the system (generally 2 or 3). The summation convention is sometimes applied to a single tensor quantity, such that

$$a_{ii} = a_{11} + a_{22} + a_{33}.$$

A tensor is invariant under a transformation of the coordinate system. For example, clockwise rotation of the coordinate system by θ about the x_3 axis can be expressed as

$$\sigma'_{mn} = R_{mi}R_{nj}\sigma_{ij}$$

where, in this case, the rotation operator $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Spatial derivatives of tensors are represented using a subscripted comma. For example,

$$\sigma_{ij,j} \equiv \frac{\partial \sigma_{ij}}{\partial x_j}.$$

Finally, time derivatives are denoted with a dot, so that $\dot{u} \equiv \frac{\partial u}{\partial t}$ and $\ddot{u} \equiv \frac{\partial^2 u}{\partial t^2}$.

specified by a unit normal vector, $\mathbf{n}$, the off-diagonal elements of the stress tensor ($i \neq$ j) represent forces applied in the plane of the face and are called *shear* stresses, whereas the diagonal elements are called *normal* stresses. In general, any stress tensor is diagonalizable and may be written in the form

$$\boldsymbol{\sigma} = \boldsymbol{\Sigma}\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1}, \tag{1.4}$$

where $\boldsymbol{\Sigma}$ is a matrix whose columns are unit eigenvectors of $\boldsymbol{\sigma}$, while $\boldsymbol{\Lambda}$ is a diagonal matrix whose elements are the corresponding eigenvalues. The eigenvectors are mutually perpendicular and are known as *principal stress* axes. These axes have particular physical significance, as they represent the normals to planes within which shear stresses vanish. The eigenvalues, called principal stresses, are denoted as σ_1, σ_2 and σ_3 and are ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. For notational reference, in the case of Cartesian coordinates these principal stresses are sometimes, but not always, equivalent to the magnitude of stress (traction) acting in the vertical direction (S_V), the maximum stress magnitude acting in a horizontal direction (S_H) and the minimum stress magnitude in a horizontal direction (S_h).

Tensor quantities can be difficult to visualize. In the case of stress tensors, a *Mohr diagram*, named in honour of its inventor, Otto Mohr, is often used to depict the state of stress (Parry, 2004). As discussed in the next chapter, the Mohr diagram also provides a useful tool to represent fault/fracture stability with respect to various failure criteria. Consider an arbitrary plane defined by unit normal $\hat{\mathbf{n}}$; for any given stress state, the traction acting on this surface can be decomposed into normal and shear components, respectively denoted as the normal stress, $\sigma_n(\hat{\mathbf{n}})$, and the shear stress, $\tau(\hat{\mathbf{n}})$. As illustrated in Figure 1.2, these two stress components serve as the coordinate axes for constructing a Mohr circle. To understand how a Mohr diagram is produced, consider the stress tensor for a simplified two-dimensional scenario

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \tag{1.5}$$

where σ_1 is the maximum principal stress and σ_2 is the minimum principal stress. With no loss of generality, a natural coordinate system is used here such that the x_1 and x_2 axes correspond to the principal stress axes, so that off-diagonal elements in the stress tensor (shear stresses) vanish. In general, the stress tensor can be expressed in a rotated coordinate system by applying a rotation transformation,

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{1.6}$$

where θ is the angle of rotation. Expanding the right side, the normal and shear stress values on the surface normal to the x'_1-axis, which makes an angle θ from the x_1 axis, are given by

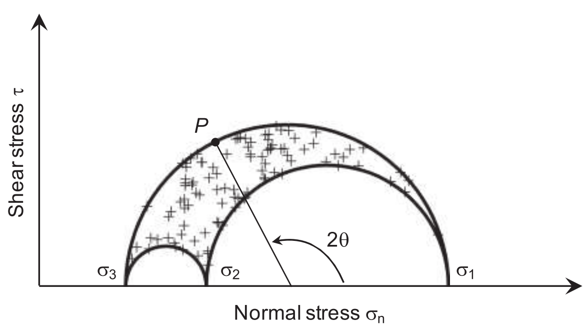

Fig. 1.2 3-D Mohr diagram, showing principal stresses, $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Symbols represent the state of stress on 100 randomly oriented fractures. Point P defines the state of stress for a plane whose normal is co-planar with the maximum and minimum principal stress axes and that makes an angle θ with respect to the maximum principal stress axis.

$$\sigma_n = \sigma'_{11} = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\left(\cos^2\theta - \sin^2\theta\right),$$
$$\tau = \sigma'_{12} = (\sigma_2-\sigma_1)\sin\theta\cos\theta = \frac{\sigma_1-\sigma_2}{2}\, 2\sin\theta\cos\theta. \tag{1.7}$$

In the expression for σ_n, we have made use of the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. By further invoking the trigonometric identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we can write

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta,$$
$$\tau = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta, \tag{1.8}$$

which are parametric equations, with respect to the variable 2θ, for a circle with centre at $\frac{\sigma_1+\sigma_2}{2}$ and radius $\frac{\sigma_1-\sigma_2}{2}$. In two dimensions, the Mohr circle thus represents the stress state as a locus of points in (σ_n, τ)-space. Each point on the circle corresponds with a plane whose normal makes an angle θ with respect to the maximum principal stress axis. A 2-D Mohr diagram is commonly represented as a semicircle by plotting with respect to $|\tau|$ rather than τ.

In three dimensions, a similar approach can be applied. First, consider the subset of normal vectors **n** that are co-planar with the maximum (σ_1) and minimum (σ_3) principal stress axes. Based on the above arguments for the 2-D case, all of the possible stress states with respect to σ_1 and τ define a circle with centre $\frac{\sigma_1+\sigma_3}{2}$ and radius $\frac{\sigma_1-\sigma_3}{2}$. Similarly, planes defined by normal vectors that are co-planar with other pairs of principal stress axes define Mohr circles with smaller radii and different centres. This set of three semicircles creates a 3-D Mohr diagram (Figure 1.2). For a given stress state defined by principal stresses σ_1, σ_2 and σ_3, it can be shown that, for all possible normal vectors, including those that are not co-planar with pairs of principal stress axes, the large Mohr circle in a 3-D Mohr diagram forms an outer boundary in (σ_n, τ)-space, whereas the smaller two Mohr circles represent inner boundaries. Referring to Figure 1.2, planes with random orientations fall within the region between the three Mohr circles.

We now turn our attention to the concept of *strain*. As shown in Figure 1.3, strain is defined with respect to an elementary volume in terms of displacement, denoted as $\mathbf{u}(\mathbf{x})$, as follows:

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1.9}$$

where the indicial comma notation for spatial derivatives has been employed on the right side of this expression. Displacement is specified here in a *Lagrangian* reference frame, which means that the coordinate system moves with a particle in the medium, consistent with seismological measurement systems (Aki and Richards, 2002). The velocity of a point in a medium is given by $\dot{\mathbf{u}}(\mathbf{x})$. Referring to Figure 1.3, it is evident that if the spatial derivatives of **u** are zero, the elementary volume is displaced with no change in shape or volume; consequently, the strain variable provides a measure of deformation in a medium. Because it is defined as a ratio of two quantities with units of length, strain is dimensionless. Strain

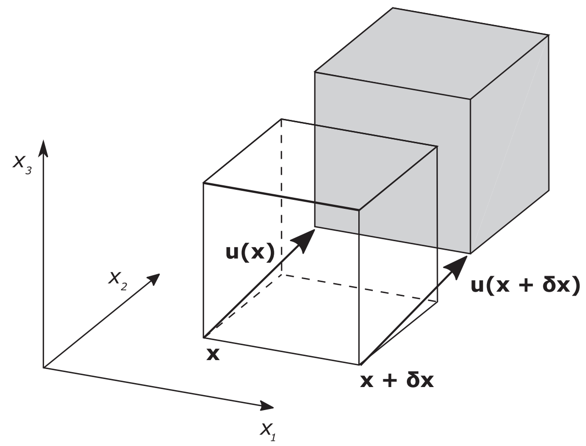

Fig. 1.3 Strain is defined by deformation of an elementary volume. The initial size and shape of the elementary volume (unshaded) are modified in response to an applied stress. The change in size and shape of the deformed volume (shaded) is described using the displacement field, $\mathbf{u}(\mathbf{x})$.

is a second-order tensor that, by definition, has the symmetry property $\epsilon_{ij} = \epsilon_{ji}$. Similar to the stress tensor, the diagonal elements are called normal-strain components and the off-diagonal elements are called shear-strain components.

1.2 Linear Elasticity

The fundamental constitutive relationship between stress and strain in a linear elastic medium is given by

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}, \tag{1.10}$$

where c_{ijkl} denotes the elastic *stiffness tensor* (recall the tensor summation convention that implies a double summation on the right side of this equation). This relationship is known as a generalized form of *Hooke's Law*. Since strain is unitless, the units of the stiffness tensor are Pa.

Since both stress and strain are second-order tensors, a fourth-order tensor (c_{ijkl}) is required to fully characterize all possible linear relationships between stress and strain components. In three dimensions, the stress tensor thus has 81 (3^4) components, where each individual scalar component is known as an *elastic modulus*. Due to various symmetries, for the general (triclinic) case the number of independent moduli can be reduced to 21. These symmetries arise from: 1) the inherent symmetry of the stress tensor ($\sigma_{ij} = \sigma_{ji}$); 2) the inherent symmetry of the strain tensor ($\epsilon_{kl} = \epsilon_{lk}$); and 3) the definition of strain-energy density, which implies that $c_{ijkl} = c_{klij}$ (Aki and Richards, 2002). A more compact notation for generalized Hooke's Law, known as *Voigt* notation, exploits these symmetries and reduces the stiffness tensor to a symmetric 6×6 stiffness matrix, $\mathbf{C}$. Using Voigt notation, the elastic constitutive relation can be expressed as

$$\begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \\ \tilde{\sigma}_4 \\ \tilde{\sigma}_5 \\ \tilde{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \tilde{C}_{14} & \tilde{C}_{15} & \tilde{C}_{16} \\ & \tilde{C}_{22} & \tilde{C}_{23} & \tilde{C}_{24} & \tilde{C}_{25} & \tilde{C}_{26} \\ & & \tilde{C}_{33} & \tilde{C}_{34} & \tilde{C}_{35} & \tilde{C}_{36} \\ & & & \tilde{C}_{44} & \tilde{C}_{45} & \tilde{C}_{46} \\ & & & & \tilde{C}_{55} & \tilde{C}_{56} \\ & & & & & \tilde{C}_{66} \end{bmatrix} \begin{bmatrix} \tilde{\epsilon}_1 \\ \tilde{\epsilon}_2 \\ \tilde{\epsilon}_3 \\ \tilde{\epsilon}_4 \\ \tilde{\epsilon}_5 \\ \tilde{\epsilon}_6 \end{bmatrix}, \tag{1.11}$$

in which pairs of indices are combined such that $()_{11} \rightarrow ()_1, ()_{22} \rightarrow ()_2, ()_{33} \rightarrow ()_3, ()_{23} \rightarrow ()_4, ()_{13} \rightarrow ()_5$ and $()_{12} \rightarrow ()_6$. For example, using this combination method $c_{1111} = \tilde{C}_{11}$ and $c_{1122} = (\tilde{C})_{12}$, where the tilde overbar notation is used here to distinguish Voigt parameters from the standard tensor representation. In addition, the Voigt strain parameters are assigned as follows: $\tilde{\epsilon}_1 = \epsilon_{11}, \tilde{\epsilon}_2 = \epsilon_{22}, \tilde{\epsilon}_3 = \epsilon_{33}, \tilde{\epsilon}_4 = 2\epsilon_{23}, \tilde{\epsilon}_5 = 2\epsilon_{13}$ and $\tilde{\epsilon}_6 = 2\epsilon_{12}$. Because $\tilde{\mathbf{C}}$ is a symmetric 6×6 matrix, this means that there is a maximum of 21 independent elastic moduli, as shown above. As a caution, it should be emphasized that the use of Voigt notation means that a more complex operator is required to transform from one coordinate system to another, known as the *Bond* transformation (see Winterstein, 1990 for details).

From an experimental perspective, the underlying mathematical model implied by the generalized form of Hooke's Law implies that 21 independent measurements of stress–strain response are required to fully characterize the elastic behaviour of a material – a daunting prospect that is seldom realized in practice. Fortunately, most rocks have inherent material symmetry properties that simplify the stress–strain relationship by reducing the number of independent coefficients needed to construct the stiffness tensor. These material symmetries arise from *rock fabric* elements that occur commonly in the subsurface, such as horizontal stratification, existence of parallel fracture sets and fabrics created by preferred alignment of minerals.

Consider the special case, albeit routinely invoked, of an *isotropic* medium. In such a medium, there is no directional dependence associated with the stress–strain relationship. Thus, for a given strain condition, a measurement of normal- or shear-stress components in a vertical orientation would yield the same result as a measurement in a horizontal orientation, or indeed at any angle of inclination. In simplistic terms, a subsurface rock mass could be considered as a fractured, fluid-saturated granular mineral aggregate. Isotropic elastic symmetry is often assumed to exist if these constituent elements, such as mineral grains or microfractures, are both small-scale and randomly oriented. Here "small scale" means small relative to the seismic wavelength, which is typically a few metres to a few hundred metres.

In the case of an isotropic medium, only two independent elastic moduli are required to fully characterize the stress–strain relationship. In this case, the elastic stiffness tensor may be written as

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right), \tag{1.12}$$

where λ and μ are independent constants known as the *Lamé* parameters and δ_{ij} is known as the *Kronecker delta*, which has the properties

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{1.13}$$

For an isotropic material, Hooke's Law may be expressed in the form

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}, \tag{1.14}$$

where the term ϵ_{kk} (implied summation over index k) is called the *dilatation*, defined as $\Delta V/V$.

Although the Lamé parameters are useful for expressing the constitutive relationship, it is often more convenient to express stiffness characteristics of a material using alternative elastic moduli that are directly linked to experimental measurements. In addition to the shear modulus, μ, other commonly used elastic moduli include *bulk modulus*, K, *Young's modulus*, E, and *Poisson's ratio*, ν. For a sample of volume V, the bulk modulus is given by

$$K \equiv -V \frac{\partial P}{\partial V}, \tag{1.15}$$

where P is confining pressure and the stress tensor has *hydrostatic* form,

$$\boldsymbol{\sigma} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}. \tag{1.16}$$

The negative sign used here reflects the convention that, whereas pressure is positive under compressional conditions, stress is positive under tensile conditions.

Young's modulus is measured under uniaxial stress conditions. For a sample of cross-sectional area A, Young's modulus can be expressed as

$$E \equiv \frac{F/A}{\Delta L/L_0} = \frac{\sigma_{axial}}{\epsilon_{axial}}, \tag{1.17}$$

where F is the force applied to the ends of the sample (positive for tensile and negative for compressional), L is the length of the sample measured along its axis in the direction of the applied force, ΔL is the change in sample length and L_0 is the sample length prior to application of the force. In addition, σ_{axial} denotes axial stress and ϵ_{axial} denotes axial strain. The shear modulus can be measured by applying a shear force to the sides of a sample and is defined as

$$\mu = \frac{\sigma_{ij}}{2\epsilon_{ij}}, \; i \neq j \text{ (no summation)}. \tag{1.18}$$

Note that in many engineering texts the shear modulus is represented by G. The elastic moduli K, E and μ for Earth materials are typically expressed in units of GPa.

Another parameter that is commonly used to describe the properties of an elastic solid is Poisson's ratio. Like Young's modulus, this is measured under uniaxial stress conditions. Poisson's ratio is unitless and is given by

$$\nu = -\frac{\epsilon_{trans}}{\epsilon_{axial}}, \tag{1.19}$$

where ϵ_{trans} is the transverse strain.