1

Mathematical Preliminaries

This preliminary chapter is dedicated to reviewing some basic mathematical results to be used in this book. The first half of this chapter is concerned with the integral transforms of Fourier, Laplace, Hankel, and Mellin and their applications to simple problems of mathematical physics with illustrative purposes. The second half contains a comprehensive review of some properties of more commonly known special functions, such as gamma, beta, and Bessel functions, followed by the presentation of other less known special functions, which arise more naturally in the framework of fractional calculus, as the Mittag-Leffler function, Wright function, and the H-function of Fox. Some integral transforms of these special functions are presented at the end of the chapter.

1.1 Integral Transforms

In mathematics, an integral transform $\mathcal{T}$ of a given function $f(t)$ has the general form

$$\mathcal{T}\{f(t);s\} := F(s) = \int_{t_1}^{t_2} K(t,s)f(t)\,dt,$$

such that the input is some function $f(t)$ and the output is another function $F(s)$. The choice of the *kernel* $K(t,s)$ defines different types of transforms. In this section, we present and review some definitions and properties of the integral transforms of Fourier, Laplace, Hankel, and Mellin, indicating a few applications to classical problems in mathematical physics. More detailed information may be found in the References at the end of the book.

1.1.1 Fourier Transforms

In the one-dimensional case, the *Fourier transform* of a function $f(x)$ of a real variable $x \in \mathbb{R} = (-\infty, \infty)$ is defined as

$$\mathcal{F}\{f(x);k\} = F(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx, \qquad k \in \mathbb{R}, \tag{1.1}$$

whereas the *inverse Fourier transform* is given by

$$\mathcal{F}^{-1}\{F(k);x\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} F(k)\,dk, \qquad k \in \mathbb{R}. \tag{1.2}$$

In Eqs. (1.1) and (1.2), $i^2 = -1$ and sometimes both are defined with a $1/\sqrt{2\pi}$ in front of the integration symbol and also with the reversed sign in the exponent. In this case, the only difference between the Fourier transform and its inverse is the signal of the exponential. The existence of the transform $F(k)$ is guaranteed if $f(x)$ is an integrable function and the integral converges. A sufficient (but not necessary) condition is to require that $f(x)$ be absolutely integrable, i.e., the integral

$$\int_{-\infty}^{\infty} |f(x)|\,dx$$

exists. In this case, $F(k)$ is absolutely convergent and, thus, it is convergent [1]. The Fourier convolution operator of two functions f and g is defined as

$$(f * g)(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau, \qquad x \in \mathbb{R}, \tag{1.3}$$

which has the commutative property, i.e.,

$$f * g = g * f.$$

The Fourier transform of the convolution (1.3) is given by the Fourier convolution theorem, which states that

$$\begin{aligned}
\mathcal{F}\{(f*g);k\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\tau)\,g(\tau)\,d\tau\right] e^{-ikx}\,dx \qquad (1.4)\\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(z)\,g(\tau)\,d\tau\right] e^{-ik(z+\tau)}\,dz\\
&= \left[\int_{-\infty}^{\infty} f(z)\,e^{-ikz}\right]\left[\int_{-\infty}^{\infty} g(\tau)\,d\tau\, e^{-ik\tau}\right] dz\\
&= F(k)G(k),
\end{aligned}$$

in which we have introduced $x = z + \tau$. Symbolically, the above expression may be rewritten as

Cambridge University Press
978-1-107-14355-5 — Fractional Diffusion Equations and Anomalous Diffusion
Luiz Roberto Evangelista , Ervin Kaminski Lenzi
Excerpt
More Information

$$\mathcal{F}\{(f * g); k\} = F(k)G(k)$$

and is well defined when the transforms $F(k)$ and $G(k)$ exist. Conversely, a useful result may be obtained when a function $W(k)$ is known in the Fourier space. If one is able to rewrite it as a product of two functions, namely, $W(k) = F(k)G(k)$, then it is possible to apply the inverse Fourier transform to (1.4), in order to obtain the identity

$$\mathcal{F}^{-1}\{F(k)G(k); x\} = (f * g)(x) = \int_{-\infty}^{\infty} f(x-\tau)\, g(\tau)\, d\tau. \tag{1.5}$$

Another useful property of the Fourier transform is the transform of the derivatives of a function $f(x)$, when $\lim_{x\to\pm\infty} f^{(p)}(x) = 0$, for $p = 0, 1, \ldots, n-1$. For the first derivative, we have

$$\mathcal{F}\left\{\frac{df}{dx}; k\right\} = \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx}\, dx.$$

After an integration by parts, it can be cast in the form

$$\begin{aligned} \mathcal{F}\left\{\frac{df}{dx}; k\right\} &= \left[ik\, e^{ikx} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, ik\, e^{ikx}\, dx \right] \\ &= -ik \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx = -ikF(k). \end{aligned} \tag{1.6}$$

Following the same procedure, the Fourier transform of the n-derivative of $f(x)$ is shown to be

$$\mathcal{F}\left\{\frac{d^n f}{dx^n}; k\right\} = (-ik)^n F(k). \tag{1.7}$$

The transform may be considered as a linear integral operator acting on a given function, such that if $f(x) = a\, g(x) + b\, h(x)$, with $(a, b) \in \mathbb{C}$, we can write

$$\begin{aligned} \mathcal{F}\{f(x); k\} &= \mathcal{F}\{a\, g(x) + b\, h(x); k\} \\ &= \int_{-\infty}^{\infty} \left[a\, g(x) + b\, h(x)\right] e^{ikx}\, dx \\ &= a \int_{-\infty}^{\infty} g(x)\, e^{ikx}\, dx + b \int_{-\infty}^{\infty} h(x)\, e^{ikx}\, dx \\ &= a\, G(k) + b\, H(k). \end{aligned} \tag{1.8}$$

Finally, we may introduce the cosine- and sine-Fourier transforms of a function $f(t)$, $t \in \mathbb{R}^+ = (0, \infty)$, defined, respectively, as

$$\mathcal{F}_c\{f(t); k\} = F_c(k) = \int_0^\infty f(t) \cos(kt)\, dt, \qquad k \in \mathbb{R}^+ \tag{1.9}$$

and

$$\mathcal{F}_s\{f(t); k\} = F_s(k) = \int_0^\infty f(t) \sin(kt)\, dt, \qquad k \in \mathbb{R}^+. \tag{1.10}$$

The Fourier transform is an important tool in many branches of mathematical physics. In particular, in linear dynamical systems, it transforms the function from the time-domain to the frequency-domain, as we discuss in Chapters 9 and 10, dedicated to some applications of fractional calculus in problems of electrochemical impedance.

Let us illustrate some of the presented results involving Fourier transforms by applying them to the classical boundary-value problem of obtaining the temperature profile $u(x, t)$ of an infinitely long bar at a point x and time t, when the initial temperature $u(x, 0) = f(x)$, such that $u(x, t) \to 0$, as $t \to \infty$. The differential equation to be solved may be written as

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty, \tag{1.11}$$

in which, for simplicity, we assume $\alpha = 1$. A trial solution of the form $u_{q,p}(x, t) = e^{px+qt}$, when substituted in (1.11), permits us to conclude that $q = p^2$ and $p = ik$, with $k \in \mathbb{R}$, such that

$$u_k(x, t) = e^{ikx} e^{-k^2 t}. \tag{1.12}$$

The initial condition $u(x, 0) = f(x)$ may be satisfied by the linear combination of solutions like (1.12), in the form

$$u(x, t) = \int_{-\infty}^{\infty} e^{ikx} e^{-k^2 t} g(k)\, dk. \tag{1.13}$$

Expression (1.13) is a solution of (1.11) provided we can differentiate under the integral sign and is valid for an arbitrary $g(k)$, which can be determined from the initial condition; i.e., since

$$u(x, 0) = f(x) = \int_{-\infty}^{\infty} e^{ikx} g(k)\, dk,$$

we have

$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx, \tag{1.14}$$

which is the inverse Fourier transform of $f(x)$. By substituting (1.14) in (1.13), we easily obtain

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 t + ik(x-x')}\, dk \right] f(x')\, dx' \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-x')^2/4t} f(x')\, dx' \\ &= \int_{-\infty}^{\infty} f(x-x') g(x') dx', \end{aligned} \tag{1.15}$$

which can also be recognised as the convolution, (1.3), in which

$$f(x-x') = \frac{1}{\sqrt{4\pi t}} e^{-(x-x')^2/4t} \quad \text{and} \quad g(x') = f(x'). \tag{1.16}$$

1.1.2 The Laplace Transform

The Laplace transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is defined as

$$\mathcal{L}\{f(t); s\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt, \qquad s \in \mathbb{C}. \tag{1.17}$$

It is an integral transform as useful as the Fourier transform in solving physical problems. For its existence, the function $f(t)$ must be such that

$$e^{-\alpha t} |f(t)| \leq M, \qquad \text{for all} \quad t \in [0, \infty),$$

where M is a positive constant and $\mathrm{Re}\,(\alpha) > 0$, indicating that the function $f(t)$ must not grow faster than a certain exponential function when $t \to \infty$. This point can be made more clearly for $s \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, because

$$\int_0^L |f(t)| e^{-st}\, dt \leq \int_0^L M e^{\alpha t} e^{-st}\, dt \leq \frac{M}{s-\alpha}.$$

The integrand remains bounded as $L \to \infty$, if $s > \alpha$. When $s \leq \alpha$, the integral diverges. Certain functions, like $f(t) = e^{t^2}$, do not have a Laplace transform.

Another feature influencing the existence of this integral transform is the presence of singularities in $f(t)$. Consider, for instance, a typical power-law behaviour $f(t) = t^n$, with $n \in \mathbb{R}$. Its Laplace transform is given by

$$\mathcal{L}\{t^n; s\} = F(s) = \int_0^\infty t^n e^{-st}\, dt, \tag{1.18}$$

which diverges for $t \to 0$, when $n \leq -1$.

The *inverse Laplace transform* is given for $t \in \mathbb{R}^+$ by the formula

$$\mathcal{L}^{-1}\{F(s); t\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds, \qquad \gamma = \mathrm{Re}\,(s) > \sigma, \tag{1.19}$$

where σ is the infimum of s values for which the Laplace integral (1.17) converges, and is called the abscissa of convergence [2]. This integral is known as the Bromwich integral, sometimes known as the Fourier–Mellin integral, as we discuss in Section 1.1.4.

The Laplace transform is linear since

$$\begin{aligned}\mathcal{L}\{a f(t) + b\, g(t); s\} &= \int_0^\infty \left[a f(t) + b\, g(t)\right] e^{-st}\, dt \\ &= a \int_0^\infty f(t)\, e^{-st}\, dt + b \int_0^\infty g(t)\, e^{-st}\, dt \\ &= a\, \mathcal{L}\{f(t); s\} + b\, \mathcal{L}\{g(t); s\} = a\, F(s) + b\, G(s). \end{aligned} \tag{1.20}$$

The Laplace convolution operator of two functions $f(t)$ and $g(t)$, given on $\mathbb{R}^+$, is defined by the integral

$$f * g = (f * g)(t) = \int_0^t f(t - x)\, g(x)\, dx = g * f, \tag{1.21}$$

which, as indicated before, has the commutative property. Now, let $F(s)$ and $G(s)$ be, respectively, the Laplace transforms of the functions $f(t)$ and $g(t)$. We can write

$$\begin{aligned} G(s)\, F(s) &= \int_0^\infty g(x)\, e^{-sx}\, dx \int_0^\infty f(y)\, e^{-sy}\, dy \\ &= \int_0^\infty dx \int_0^\infty g(x) f(y)\, e^{-s(x+y)}\, dy \end{aligned}$$

$$= \int_{y=0}^{\infty} dy \int_{t=y}^{\infty} g(t-y) f(y)\, e^{-st}\, dt$$

$$= \mathcal{L}\left\{ \int_{0}^{t} g(t-y) f(y)\, dy; s \right\}, \tag{1.22}$$

where we have changed the variables by inserting $t = x + y$. In Eq. (1.22), we recognise the definition (1.21), such that

$$G(s)F(s) = \int_{0}^{\infty} \left[(g * f)(t)\right] e^{-st}\, dt = \mathcal{L}\{(g*f)(t); s\}$$
$$= \mathcal{L}\{g(t); s\}\, \mathcal{L}\{f(t); s\}\,.$$

Some formulas for the Laplace transform of elementary and generalised function are as follows. If $f(t) = 1$, the Laplace transform is

$$\mathcal{L}\{1; s\} = \int_{0}^{\infty} 1\, e^{-st}\, dt = \frac{1}{s}.$$

For $f(t) = t^{\alpha}$, with $\mathrm{Re}\,(\alpha) > -1$, we have

$$\mathcal{L}\{t^{\alpha}; s\} = \int_{0}^{\infty} t^{\alpha}\, e^{-st}\, dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \qquad \mathrm{Re}\,(s) > 0.$$

In particular, if $\alpha = k \in \mathbb{N}_0 = \{0, 1, 2, \ldots,\}$, we obtain

$$\mathcal{L}\{t^{k}; s\} = \frac{k!}{s^{k+1}}. \tag{1.23}$$

For $f(t) = t^{-1-k}$, we can show that

$$\mathcal{L}\{t^{-1-k}; s\} = \int_{0}^{\infty} t^{-1-k}\, e^{-st}\, dt$$
$$= (-1)^{k+1} \frac{s^{k}}{k!} \left[\ln(s) - \psi(k+1)\right],$$

with $\psi(k)$ being the Euler's psi function, i.e., the logarithm derivative of the gamma function, to be defined in Eq. (1.85).

The Laplace transform of the Dirac delta function is

$$\mathcal{L}\{\delta(t); s\} = \int_{0}^{\infty} \delta(t)\, e^{-st}\, dt = 1, \qquad s \in \mathbb{C}.$$

For the derivatives, we simply have

$$\mathcal{L}\{\delta^k(t);s\} = s^k, \qquad s \in \mathbb{C} \quad \text{and} \quad k \in \mathbb{N}.$$

Finally, the Laplace transform of a derivative of a function $f(t)$ is given by

$$\mathcal{L}\left\{\frac{df}{dt};s\right\} = -e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)\,e^{-st}\,dt = -f(0) + s\,F(s), \tag{1.24}$$

where an integration by parts has been done. This result may be generalised for the k-derivative, as

$$\mathcal{L}\left\{\frac{d^kf}{dt^k};s\right\} = s^kF(s) - \sum_{p=0}^{k-1} s^p \frac{d^{(k-p-1)}}{dt^{(k-p-1)}}f(t)\Big|_{t=0}. \tag{1.25}$$

For those functions such that $f(0) = 0$, we obtain the very useful result:

$$\mathcal{L}\left\{\frac{d^kf}{dt^k};s\right\} = s^kF(s).$$

The Laplace transform method is particularly suitable for studying wave propagation along transmission lines and physical problems with boundary conditions involving time derivatives. One illustration of the use of the Laplace transform method may be worked out from the same example treated in the previous section, i.e., to solve the problem:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x), \quad u(x,t) \to 0, \quad t \to \infty. \tag{1.26}$$

Since $u(x,t)$ is assumed as bounded, $|u(x,t)| \leq M$, the Laplace transform exists in the form

$$U(x,s) = \int_0^\infty e^{-st}u(x,t)dt, \tag{1.27}$$

such that

$$|U(x,s)| \leq \int_0^\infty e^{-st}\,|u(x,t)|dt \leq \frac{M}{s}.$$

Applying the Laplace transform to the first of Eqs. (1.26), we obtain:

$$\frac{d^2U}{dx^2} = sU(x,s) - u(x,0) = sU(x,s) - f(x), \tag{1.28}$$

Cambridge University Press
978-1-107-14355-5 — Fractional Diffusion Equations and Anomalous Diffusion
Luiz Roberto Evangelista , Ervin Kaminski Lenzi
Excerpt
More Information

where we have used (1.24) and the initial condition in (1.26). Equation (1.28) may be rewritten as

$$\frac{d^2U}{dx^2} - sU(x,s) = -f(x), \tag{1.29}$$

and its general solution has the form

$$U(x,s) = U_h(x,s) + U_p(x,s), \tag{1.30}$$

where

$$U_h(x,s) = c_1 U_1(x,s) + c_2 U_2(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} \tag{1.31}$$

is the general solution of the homogeneous problem, whereas $U_p(x,s)$ is any particular solution of the nonhomogenous problem. The Wronskian may be calculated as

$$W(U_1, U_2) = \begin{vmatrix} U_1(x,s) & U_2(x,s) \\ U_1'(x,s) & U_2'(x,s) \end{vmatrix} = -2\sqrt{s}.$$

The particular solution is thus

$$\begin{aligned} U_p(x,s) &= \int_0^x \frac{-U_1(x,s)U_2(y,s) + U_2(x,s)U_1(y,s)}{W(y,s)} \left[-f(y)\right] dy \\ &= \frac{e^{\sqrt{s}x}}{2\sqrt{s}} \int_0^x e^{-\sqrt{s}y} f(y)dy + \frac{e^{-\sqrt{s}x}}{2\sqrt{s}} \int_0^x e^{\sqrt{s}y} f(y)dy. \end{aligned} \tag{1.32}$$

The general solution (1.30) may be written as

$$U(x,s) = e^{\sqrt{s}x}\left[c_1 - \frac{1}{2\sqrt{s}}u_-(s)\right] + e^{-\sqrt{s}x}\left[c_2 + \frac{1}{2\sqrt{s}}u_+(s)\right], \tag{1.33}$$

where

$$u_\pm(s) = \int_0^x e^{\pm\sqrt{s}y} f(y)dy.$$

Since the solution (1.33) has to stay bounded when $|x| \to \infty$, we require that

$$\lim_{x\to\infty}\left[c_1 - \frac{1}{2\sqrt{s}}\int_0^x e^{-\sqrt{s}y} f(y)dy\right] = 0$$

and

$$\lim_{x\to-\infty}\left[c_2+\frac{1}{2\sqrt{s}}\int_0^x e^{+\sqrt{s}y}f(y)dy\right]=0,$$

which implies

$$c_1=\frac{1}{2\sqrt{s}}\int_0^\infty e^{-\sqrt{s}y}f(y)dy \quad \text{and} \quad c_2=\frac{1}{2\sqrt{s}}\int_{-\infty}^0 e^{\sqrt{s}y}f(y)dy. \tag{1.34}$$

The solution (1.33) becomes:

$$\begin{aligned} U(x,s) &= \frac{e^{\sqrt{s}x}}{2\sqrt{s}}\int_x^\infty e^{-\sqrt{s}y}f(y)dy+\frac{e^{-\sqrt{s}x}}{2\sqrt{s}}\int_{-\infty}^x e^{\sqrt{s}y}f(y)dy \\ &= \frac{1}{2\sqrt{s}}\int_{-\infty}^\infty e^{-\sqrt{s}|x-y|}f(y)dy. \end{aligned} \tag{1.35}$$

The problem is formally solved in the Laplace space (s-domain). To find the solution in the t-domain, we have to obtain the inverse Laplace transform of (1.35), i.e.,

$$u(x,t)=\mathcal{L}^{-1}\{U(x,s);t\}=\int_{-\infty}^\infty \mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}|x-y|}}{2\sqrt{s}};t\right\}f(y)dy. \tag{1.36}$$

If we put $a=|x-y|$, from the tables of Laplace transforms [3] we obtain

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{2\sqrt{s}};t\right\}=\frac{e^{-a^2/4t}}{\sqrt{\pi t}}.$$

Finally, the solution may be written as

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^\infty e^{-(x-y)^2/4t}f(y)dy, \tag{1.37}$$

which coincides, obviously, with the result stated in (1.15).

1.1.3 The Hankel Transform

The Hankel transform (or Fourier–Bessel integral) is appropriate for those problems in which there is axial symmetry and the radial variable ranges from 0 to ∞. It is defined as