PASSIVE IMAGING WITH AMBIENT NOISE

Waves generated by opportunistic or ambient noise sources and recorded by passive sensor arrays can be used to image the medium through which they travel. Spectacular results have been obtained in seismic interferometry, which open up new perspectives in acoustics, electromagnetics, and optics. The authors present, for the first time in book form, a self-contained and unified account of correlation-based and ambient noise imaging. In order to facilitate understanding of the core material, they also address a number of related topics in conventional sensor array imaging, wave propagation in random media, and high-frequency asymptotics for wave propagation. Taking a multidisciplinary approach, the book uses mathematical tools from probability, partial differential equations, and asymptotic analysis, combined with the physics of wave propagation and modeling of imaging modalities. Suitable for applied mathematicians and geophysicists, it is also accessible to graduate students in applied mathematics, physics, and engineering.

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PASSIVE IMAGING WITH AMBIENT NOISE

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# Contents

## Preface

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction and overview of the book</td>
<td>xi</td>
</tr>
<tr>
<td>1.1 Why passive, correlation-based imaging?</td>
<td>1</td>
</tr>
<tr>
<td>1.1.1 Travel time estimation</td>
<td>2</td>
</tr>
<tr>
<td>1.1.2 Applications of travel time estimation</td>
<td>3</td>
</tr>
<tr>
<td>1.1.3 Reflector imaging</td>
<td>4</td>
</tr>
<tr>
<td>1.1.4 Auxiliary array or virtual source imaging</td>
<td>6</td>
</tr>
<tr>
<td>1.1.5 Passive synthetic aperture imaging</td>
<td>8</td>
</tr>
<tr>
<td>1.1.6 Imaging with intensity cross correlations</td>
<td>9</td>
</tr>
<tr>
<td>1.2 Chapter-by-chapter description of the book</td>
<td>10</td>
</tr>
</tbody>
</table>

## 2 Green’s function estimation from noise cross correlations

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 The scalar wave equation and its Green’s function</td>
<td>17</td>
</tr>
<tr>
<td>2.1.1 The Sommerfeld radiation condition</td>
<td>19</td>
</tr>
<tr>
<td>2.1.2 Reciprocity</td>
<td>20</td>
</tr>
<tr>
<td>2.1.3 The Helmholtz-Kirchhoff identity</td>
<td>21</td>
</tr>
<tr>
<td>2.1.4 Application to time reversal</td>
<td>23</td>
</tr>
<tr>
<td>2.2 The scalar wave equation with noise sources</td>
<td>25</td>
</tr>
<tr>
<td>2.3 Green’s function estimation with a uniform distribution of sources in a homogeneous open medium</td>
<td>29</td>
</tr>
<tr>
<td>2.4 Green’s function estimation with an extended distribution of sources in an inhomogeneous open medium</td>
<td>31</td>
</tr>
<tr>
<td>2.5 Green’s function estimation with an extended distribution of sources in an inhomogeneous cavity</td>
<td>34</td>
</tr>
<tr>
<td>2.6 Green’s function estimation with a limited distribution of sources in a one-dimensional inhomogeneous medium</td>
<td>38</td>
</tr>
<tr>
<td>2.6.1 The one-dimensional wave equation</td>
<td>39</td>
</tr>
</tbody>
</table>
Contents

2.6.2 Reflection seismology 43
2.6.3 Daylight imaging 45
2.7 Conclusion 48
2.8 Appendix: the covariance of the empirical cross correlation 48

3 Travel time estimation from noise cross correlations using stationary phase 51
3.1 High-frequency wave propagation 52
3.2 High-frequency asymptotic analysis of the Green’s function in a homogeneous medium 53
3.3 High-frequency asymptotic analysis of the Green’s function in a smoothly varying medium 53
3.3.1 An introduction to geometrical optics 53
3.3.2 Ray solution of the eikonal equation 55
3.3.3 Fermat’s principle for the travel time 57
3.3.4 Properties of the travel time 58
3.4 High-frequency asymptotic analysis of the cross correlation 60
3.5 Conclusion 67

4 Overview of conventional sensor array imaging 68
4.1 Passive array imaging of sources 68
4.1.1 Data acquisition 68
4.1.2 Imaging function 69
4.1.3 The linear forward operator 69
4.1.4 The adjoint operator 70
4.1.5 Least squares inversion 71
4.1.6 The reverse-time imaging function 73
4.1.7 Kirchhoff migration (or travel-time migration) 74
4.2 Passive array imaging of sources: resolution analysis 74
4.2.1 Full-aperture array 75
4.2.2 Partial-aperture array 75
4.2.3 Summary of resolution analysis for passive source imaging 83
4.3 Active array imaging of reflectors 84
4.3.1 Data acquisition 84
4.3.2 Source and reflector array imaging: comparison 85
4.3.3 Modeling 85
4.3.4 Nonlinear inversion 86
4.3.5 Linearization of the forward problem 86
4.3.6 Linearized inversion 88
4.3.7 The reverse-time imaging function 89
4.3.8 Kirchhoff migration (or travel-time migration) 91
4.3.9 Summary of resolution analysis for active reflector imaging 91
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.4 A remark about time-reversal experiments</td>
<td>92</td>
</tr>
<tr>
<td>4.5 Conclusion</td>
<td>92</td>
</tr>
<tr>
<td>5 Passive array imaging of reflectors using ambient noise illumination</td>
<td>94</td>
</tr>
<tr>
<td>5.1 Imaging configurations of noise sources, sensors, and reflectors</td>
<td>94</td>
</tr>
<tr>
<td>5.2 Stationary phase analysis of the cross correlation with reflectors</td>
<td>96</td>
</tr>
<tr>
<td>5.3 Migration imaging of cross correlations</td>
<td>99</td>
</tr>
<tr>
<td>5.3.1 Migration imaging with daylight illumination</td>
<td>100</td>
</tr>
<tr>
<td>5.3.2 Migration imaging with backlight illumination</td>
<td>101</td>
</tr>
<tr>
<td>5.3.3 Migration imaging with surround light illumination</td>
<td>103</td>
</tr>
<tr>
<td>5.4 Conclusion</td>
<td>105</td>
</tr>
<tr>
<td>6 Resolution analysis for passive array imaging using ambient noise illumination</td>
<td>106</td>
</tr>
<tr>
<td>6.1 A comparison of reflector imaging with active and passive arrays</td>
<td>107</td>
</tr>
<tr>
<td>6.2 Imaging by cross correlation of signals generated by ambient noise sources</td>
<td>108</td>
</tr>
<tr>
<td>6.2.1 The wave equation with noise sources</td>
<td>108</td>
</tr>
<tr>
<td>6.2.2 Statistical stability of the cross correlation function</td>
<td>108</td>
</tr>
<tr>
<td>6.2.3 Passive sensor imaging</td>
<td>109</td>
</tr>
<tr>
<td>6.2.4 Hypothesis of small decoherence time and correlation radius for the noise sources</td>
<td>110</td>
</tr>
<tr>
<td>6.3 Structure of the cross correlations in a homogeneous medium</td>
<td>111</td>
</tr>
<tr>
<td>6.3.1 The background Green’s function</td>
<td>111</td>
</tr>
<tr>
<td>6.3.2 The peaks of the cross correlation in the presence of a reflector</td>
<td>111</td>
</tr>
<tr>
<td>6.4 Resolution analysis of correlation-based imaging</td>
<td>115</td>
</tr>
<tr>
<td>6.4.1 The daylight imaging function</td>
<td>115</td>
</tr>
<tr>
<td>6.4.2 The backlight imaging function</td>
<td>122</td>
</tr>
<tr>
<td>6.4.3 Numerical simulations</td>
<td>124</td>
</tr>
<tr>
<td>6.4.4 Role of illumination diversity</td>
<td>125</td>
</tr>
<tr>
<td>6.5 Conclusion</td>
<td>126</td>
</tr>
<tr>
<td>6.A Appendix: Proof of Proposition 6.2</td>
<td>126</td>
</tr>
<tr>
<td>6.B Appendix: Proof of Propositions 6.4–6.5</td>
<td>128</td>
</tr>
<tr>
<td>6.C Appendix: Proof of Proposition 6.6</td>
<td>132</td>
</tr>
<tr>
<td>7 Travel time estimation using ambient noise in weakly scattering media</td>
<td>136</td>
</tr>
<tr>
<td>7.1 Role of scattering in travel time estimation with cross correlations</td>
<td>136</td>
</tr>
<tr>
<td>7.2 A model for the scattering medium</td>
<td>138</td>
</tr>
<tr>
<td>7.3 Signal-to-noise ratio reduction and enhanced resolution due to scattering</td>
<td>140</td>
</tr>
<tr>
<td>7.4 Use of fourth-order cross correlations</td>
<td>142</td>
</tr>
</tbody>
</table>
## Contents

7.5 Conclusion 145  
7.A Appendix: Complete expression of the average cross correlation 146  
7.B Appendix: Proof of Proposition 7.1 148  
7.C Appendix: Proof of Proposition 7.2 149  

8 Correlation-based reflector imaging using ambient noise in weakly scattering media 152  
8.1 Role of scattering in correlation-based imaging 152  
8.2 Passive sensor imaging in a randomly scattering medium 154  
8.2.1 A model for the scattering medium 155  
8.2.2 The differential cross correlation 156  
8.2.3 Expansion of the clutter Green’s function 157  
8.2.4 Expansion of the differential cross correlation 159  
8.2.5 Statistical analysis of the differential cross correlation 160  
8.2.6 On the trade-off between resolution enhancement and signal-to-noise ratio reduction 164  
8.2.7 Numerical simulation of migration imaging with cross correlations in the presence of scatterers 164  
8.3 Passive sensor imaging with a reflecting interface 165  
8.3.1 Stationary phase analysis of the cross correlation with a reflecting interface 166  
8.3.2 Numerical simulations of migration imaging with cross correlations in the presence of an interface 168  
8.4 Iterated cross correlations for passive imaging in a randomly scattering medium 170  
8.4.1 The coda cross correlation 170  
8.4.2 Numerical simulations of migration imaging with coda cross correlations 172  
8.5 Conclusion 172  
8.A Appendix: Proof of Proposition 8.1 174  
8.B Appendix: Proof of Proposition 8.2 178  
8.B.1 First group 178  
8.B.2 Second group 180  
8.C Appendix: Statistical analysis of the cross correlations 182  
8.C.1 The cross correlation at the difference of travel times 182  
8.C.2 The cross correlation at the sum of travel times 184  
8.D Appendix: Proof of Proposition 8.3 185  

9 Virtual source imaging in homogeneous media 187  
9.1 Introduction to virtual source imaging 187  
9.2 Ideal virtual source imaging with an infinite source array 190
Contents ix

9.3 High-frequency analysis in a homogeneous background with a limited source array 191
  9.3.1 Direct scattering problem 191
  9.3.2 High-frequency analysis of the cross correlations 192
  9.3.3 High-frequency analysis of the imaging function 195

9.4 Passive synthetic aperture imaging in a homogeneous background 197
  9.4.1 High-frequency analysis of the imaging function 198
  9.4.2 Comparison with classical synthetic aperture imaging 199

9.5 Conclusion 201


10 Virtual source imaging in scattering media 206
  10.1 The auxiliary array imaging setup 206
  10.2 Time-reversal interpretation of virtual source imaging 208
  10.3 The paraxial approximation in random media 209
    10.3.1 The main results in the paraxial approximation 210
    10.3.2 Validity of the paraxial approximation in random media 211
  10.4 Analysis of virtual source imaging in the random paraxial regime 212
    10.4.1 The cross correlation of the recorded field 212
    10.4.2 Migration of cross correlations 216
  10.5 Numerical simulations 218
  10.6 Passive synthetic aperture imaging in random media 219
  10.7 Conclusion 222
  10.A Appendix: Proofs of Propositions 10.1–10.2 223

11 Imaging with intensity cross correlations 228
  11.1 The ghost imaging setup 228
  11.2 The intensity correlation function 231
    11.2.1 The empirical and statistical correlations 231
    11.2.2 Paraxial regime 233
    11.2.3 Time-reversal interpretation 235
    11.2.4 Averaging with respect to the random medium 236
  11.3 Resolution analysis 237
    11.3.1 Resolution analysis for the fully incoherent case 237
    11.3.2 Resolution analysis for the partially coherent case 240
  11.4 Conclusion 242
  11.A Appendix: The fields in the white-noise paraxial regime 243

12 A review of wave propagation in random media 245
  12.1 The random travel time model 245
## Contents

12.1.1 Domain of validity 245  
12.1.2 Statistics of the amplitude and phase perturbations 247  
12.1.3 The moments of the Green’s function 250

12.2 The random paraxial model 253  
12.2.1 The random paraxial regime 253  
12.2.2 The random paraxial wave equation 254  
12.2.3 The moments of the fundamental solution 255

12.3 The randomly layered model 258  
12.3.1 The scaling regime 258  
12.3.2 Review of wave propagation in randomly layered media 260  
12.3.3 Statistics of the Green’s function 261

12.4 Conclusion 262

12.A Appendix: Proof of Lemma 12.1 262  
12.B Appendix: Proof of Proposition 12.6 264  
12.C Appendix: Proof of Proposition 12.8 267

13 Appendix: Basic facts from analysis and probability 269  
13.1 Fourier identities 269  
13.2 Divergence theorem 270  
13.3 Stationary phase method 270  
13.4 Sampling theorem 272  
13.5 Random processes 274  
13.5.1 Random variables 274  
13.5.2 Random vectors 275  
13.5.3 Gaussian random vectors 276  
13.5.4 Random processes 277  
13.5.5 Ergodic processes 278  
13.5.6 Mean square theory 279  
13.5.7 Gaussian processes 281  
13.5.8 Stationary Gaussian processes 282  
13.5.9 Vector- and complex-valued Gaussian processes 283

References 285  
Index 293
Preface

In sensor array imaging the objective is to probe an unknown medium with waves, and this can be done in two steps. In the data acquisition step, waves are emitted by a source array, they propagate through the medium being probed, and they are recorded by a receiver array. In the data processing step, this data set is used to extract information about the medium, such as the location of reflectors embedded in it. Recent developments in imaging in complex media and on passive, ambient noise imaging have had a profound impact in many different applied fields. It is these developments that have motivated us to write this book.

Research in wave propagation in complex, scattering media has been active for a long time. It is relevant in many imaging applications in which one wants to probe a medium such as the Earth’s lithosphere in seismic imaging, concrete structures in non-destructive testing, the human body in medical imaging, the turbulent atmosphere in optical imaging, or shallow water environments in acoustic imaging. These media are quite complex, while we often want to image only some particular features in them. It turns out that the established imaging methods, which we also describe in this book, may fail when the ambient medium is scattering. It is only recently that wave propagation in complex media, modeled by random media, has been formulated and analyzed in a way that makes it possible to develop new imaging techniques that can mitigate the effects of random scattering.

Array imaging has also been analyzed and used in many applications for a long time. The recent trend to deploy large sensor arrays is due to improved sensor technology, reduced cost in data storage, and increased computational capabilities. In particular, passive sensor array imaging has recently become an area of intense research activity because of the potential impact of its applications. Passive means here that only receiver arrays are used, instead of active source/receiver arrays, and the illumination is provided by unknown, uncontrolled, asynchronous, or opportunistic sources. Imaging with ambient noise sources is one of the main topics of this book. Of course, the structure of the array data is quite different from that of active sensor arrays, and this requires the development of new imaging techniques.

Both in the field of imaging in complex media and in passive, ambient noise imaging, the theoretical analysis shows that the cross correlations between the recorded signals play
Preface

an important role. This is because they carry information about the medium through which the waves propagate, and they also reduce spurious noise effects. Interferometric imaging is another commonly used term for correlation-based imaging. The study of the wave field correlations is central to this book. In seismology the emergence of correlation-based imaging using ambient seismic noise has had a profound impact. The use of seismograms generated by earthquakes was previously the only way to image the Earth. With correlation-based imaging, the apparent seismic noise recorded by a distributed network of sensors on the Earth’s surface can provide a lot of information about its structure. Beyond seismology, there are many new, emerging areas for correlation-based imaging methods, in passive synthetic aperture radar or in optical speckle intensity correlations for communications and imaging, as discussed in the last chapters of the book.

The subject of this book is multidisciplinary. It uses mathematical tools from probability and stochastic processes, partial differential equations and asymptotic analysis, combined with the physics of wave propagation and modeling of imaging modalities in complex environments. However, the essential results can be obtained at an elementary level by using the multidimensional stationary phase method. This book is aimed at readers with interdisciplinary interests, and in particular students and researchers engaged with imaging methodologies related to wave propagation and sensor arrays.

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