Complex Analysis

This user-friendly textbook introduces complex analysis at the beginning graduate or advanced undergraduate level. Unlike other textbooks, it follows Weierstrass's approach, stressing the importance of power series expansions instead of starting with the Cauchy integral formula, an approach that illuminates many important concepts. This view allows readers to quickly obtain and understand many fundamental results of complex analysis, such as the maximum principle, Liouville's theorem and Schwarz's lemma.

The book covers all the essential material on complex analysis, and includes several elegant proofs that were recently discovered. It includes the zipper algorithm for computing conformal maps, a constructive proof of the Riemann mapping theorem, and culminates in a complete proof of the uniformization theorem. Aimed at students with some undergraduate background in real analysis, though not Lebesgue integration, this classroom-tested textbook will teach the skills and intuition necessary to understand this important area of mathematics.

Donald E. Marshall is Professor of Mathematics at the University of Washington. He received his PhD from UCLA in 1976. Professor Marshall is a leading complex analyst with a very strong research record that has been continuously funded throughout his career. He has given invited lectures in over a dozen countries. He is coauthor of the research-level monograph *Harmonic Measure*, published by Cambridge University Press.

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Donald E. Marshall
Frontmatter
More Information

Contents

Figures

Preface

This book provides a graduate-level introduction to complex analysis. There are four points of view for this subject due primarily to Cauchy, Weierstrass, Riemann and Runge. Cauchy thought of analytic functions in terms of a complex derivative and through his famous integral formula. Weierstrass instead stressed the importance of power series expansions. Riemann viewed analytic functions as locally rigid mappings from one region to another, a more geometric point of view. Runge showed that analytic functions are nothing more than limits of rational functions. The seminal modern text in this area was written by Ahlfors [1], which stresses Cauchy's point of view. Most subsequent texts have followed his lead. One aspect of the first-year course in complex analysis is that the material has been around so long that some very slick and elegant proofs have been discovered. The subject is quite beautiful as a result, but some theorems then may seem mysterious.

I have decided instead to start with Weierstrass's point of view for local behavior. Cartan [4] has a similar approach. Power series are elementary and give you many non-trivial functions immediately. In many cases it is a lot easier to see why certain theorems are true from this point of view. For example, it is remarkable that a function which has a complex derivative actually has derivatives of all orders. However, the derivative of a power series is just another power series and hence has derivatives of all orders.

Cauchy's theorem is a more global result concerned with integrals of analytic functions. Why integrals of the form $\int \frac{1}{z-a}dz$ are important in Cauchy's theorem is very easy to understand using partial fractions for rational functions. So we will use Runge's point of view for more global results: analytic functions are simply limits of rational functions.

As a pedagogic device we will use the term "analytic" for local power series expansion and "holomorphic" for possessing a continuous complex derivative. We will of course prove that these concepts (and several others) are equivalent eventually, but in the early chapters the reader should be alert to the different definitions.

The emphasis in Chapters 1–6 is to view analytic functions as behaving like polynomials or rational functions. Perhaps the most important elementary tool in this subject is the maximum principle, highlighted in Chapter 3. Runge's theorem is proved in Chapter 4 and is used to prove Cauchy's theorem in Chapter 5. Chapter 6 uses color to visualize complex-valued functions. Given a coloring of the complex plane, a function f can be illustrated by placing the color of $f(z)$ at the point z. See Section A.2 of the appendix for a computer program to do this.

Chapters 7 and 8 introduce harmonic and subharmonic functions and highlight their application to the study of analytic functions. Chapter 8 includes a method, called the geodesic zipper algorithm, for numerically computing conformal maps, which is fast and simple to program. Together with Harnack's principle, it is used to give a somewhat constructive proof of the Riemann mapping theorem in Chapter 8. Because it does not require the development of normal families, it is possible to give a one-quarter course that includes this proof of the Riemann mapping theorem. The standard proof based on normal families is given in Chapter 10. In Chapter 10 we also give Zalcman's remarkable characterization of non-normal families in terms of an associated convergent sequence, then use it to prove Montel's theorem and Picard's great theorem.

Complete and accessible proofs of Carathéodory's theorem and the Jordan curve theorem are included in Chapter 12. Local barriers instead of barriers are used to analyze regular points for the Dirichlet problem in Chapter 13, so that it is easier to verify that every boundary point of a simply-connected region is regular. This allows us to give another proof of the Riemann mapping theorem. The uniformization theorem and the classification of all Riemann surfaces in Chapters 14–16 tie together complex functions, topology, manifolds and groups. The proof of the uniformization theorem here uses Green's function, when available, and the dipole Green's function otherwise. This yields a very similar treatment of the two cases. The main tool is simply the maximum principle, which allows a proof that avoids the "oil speck" method of exhaustion by relatively compact surfaces, and avoids the need to prove triangulation or Green's theorem on Riemann surfaces. Another benefit of this approach is that it is then easy to construct plenty of meromorphic functions on any Riemann surface in Chapter 16. The first section in the appendix lists 15 ways developed in the text to determine whether a function is analytic.

Each of the three parts of this book can be comfortably covered in a one-quarter course. A one-semester course might include most of the material in Chapters 1–9. A list of prerequisites follows this preface. Students should be encouraged to review this material as needed, especially if they encounter difficulties in Chapter 1. Lebesgue integration is not needed in this text because, by Theorem 4.32, we can integrate an analytic function on any continuous curve using Riemann integration. The exercises at the end of each chapter are divided by difficulty, though in some cases they can be solved in more than one way. Exercises A are mostly straightforward, requiring little originality, and are designed for practice with the material. The B exercises require a good idea or non-routine use of the results in the chapter. Sometimes a creative idea or the right insight can lead to a simple solution. The C exercises are usually much more difficult. You can think of "C" as "challenge." I generally ask students to do the A exercises while reading, but focus on the B exercises for homework. Class discussions are facilitated by asking the students to read as much as they can before we discuss the material. Most of the B exercises come from the PhD qualifying exams in complex analysis at the University of Washington [18]. It is entirely possible to find solutions to problems by searching the internet. It is also possible to solve some problems using more advanced techniques or theorems than have been covered in the text. Both will defeat the purpose of developing the ability to solve problems, a goal of this book. For that reason, we also ask you not to tempt others by posting solutions.

This book is not written as a novel that can be read passively. Active involvement will increase your understanding as you read this material. You should have plenty of scratch paper at hand so that you can check all details. The ideas in a proof are at least as important as the statement of the corresponding theorem, if not more so. But the ideas are meaningful only if you can fill in all the details. View this as practice for proving your own theorems.

I am grateful to many people for their assistance in preparing this text. First are all of the students in my complex analysis classes who have pointed out errors, omissions and less than stellar explanations. But I would particularly like to thank John Garnett, Pietro Poggi-Corradini and Steffen Rohde, who have used the material in their classes and made numerous excellent suggestions for improvement. Similar thanks go to Robert Burckel, who read the first thirteen chapters with the eye of an eagle and a fine-toothed comb. I owe a great deal to all my teachers, coauthors and the books I have read for the mathematics they have taught me. Hopefully, some of the elegance, beauty and technique have been retained here. Several excellent texts on this subject are also listed in the bibliography. As with any mathematics text, errors still no doubt remain. I would appreciate receiving email at dmarshal@uw.edu about any errors you encounter. I will list corrections on the web page:
www.cambridge.org/marshall

Wov'n through carefully chosen words
Lie the strands of key support
By my truelove Marianne
Mainstay of our life journey
Fifty years with more to come.

Prerequisites

You should be on friendly terms with the following concepts. If you have only seen the corresponding proofs for real numbers and real-valued functions, check to see whether the same proofs also work when "real" is replaced by "complex," after reading the first two sections of Chapter 1. As you read the text, check all the details. If many of the concepts below are new to you, then I would recommend that you first take a senior-level analysis class.

Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of real numbers and let $\{f_n\}$ be a sequence of real-valued, continuous functions defined on some interval $I \subset \mathbb{R}$.

1. $\{a_n\}$ converges to a (notation: $a_n \to a$) "$\epsilon - \delta$" version.
2. Cauchy sequence.
3. $\sum a_n$ converges, converges absolutely (notation: $\sum |a_n| < \infty$).
4. $\sum a_n$ converges implies $a_n \to 0$, but not conversely;
5. $\limsup_{n\to\infty} a_n$, $\liminf_{n\to\infty} a_n$.
6. Comparison test for convergence.
7. Rearranging absolutely convergent series gives the same sum, but a similar statement does not hold for for conditionally convergent series.
8. If $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$ then

$$A + B = \sum_{n=0}^{\infty}(a_n + b_n) \quad \text{and} \quad cA = \sum_{n=0}^{\infty} ca_n.$$

If $\sum a_n$ converges absolutely and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ then

$$AB = \sum_{n=0}^{\infty} c_n.$$

9. $$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{n,k}$$

provided at least one sum converges absolutely. Absolute convergence of either of these double sums is equivalent to the finiteness of

$$\sup_{S \text{ finite}} \sum_{n,k\in S} |a_{n,k}|.$$

10. Continuous function, uniformly continuous function.
11. $f_n(x) \to f(x)$ pointwise, $f_n(x) \to f(x)$ uniformly.
12. Uniform limit of a sequence of continuous functions is continuous.

13. $\left|\int_I f(x)dx\right| \le \int_I |f(x)|dx$.
14. If $f_n \to f$ uniformly on a bounded interval I then

$$\lim \int_I f_n(x)dx = \int_I \lim f_n(x)dx = \int_I f(x)dx.$$

15. Corollary:

$$\sum_{n=0}^{\infty} \int_I f_n(x)dx = \int_I \sum_{n=0}^{\infty} f_n(x)dx,$$

    if the partial sums of $\sum f_n$ converge uniformly on the bounded interval I.
16. Open set, closed set, connected set, compact set, metric space.
17. f continuous on a compact set X implies f is uniformly continuous on X.
18. $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.
19. A metric space X is compact if and only if every infinite sequence in X has a limit (cluster) point in X. (This can fail if X is not a metric space.)
20. If f is continuous on a connected set U then $f(U)$ is connected. If f is continuous on a compact set K then $f(K)$ is compact.
21. A continuous real-valued function on a compact set has a maximum and a minimum.

All of the above can be found in the undergraduate text Rudin [22], as well as many other sources.