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PART I

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1 Preliminaries

1.1 Complex Numbers

The **complex numbers** $\mathbb{C}$ consist of pairs of real numbers: $\{(x, y) : x, y \in \mathbb{R}\}$. The complex number (x, y) can be represented geometrically as a point in the plane $\mathbb{R}^2$, or viewed as a vector whose tip has coordinates (x, y) and whose tail has coordinates $(0, 0)$. The complex number (x, y) can be identified with another pair of real numbers (r, θ), called the polar coordinate representation. The line from $(0, 0)$ to (x, y) has length r and forms an angle θ with the positive x axis. The angle is measured by using the distance along the corresponding arc of the circle of radius 1 (centered at $(0, 0)$). By similarity, the length of the subtended arc on the circle of radius r is $r\theta$. See Figure 1.1.

Conversion between these two representations is given by

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Care must be taken to find θ from the last equality since many angles can have the same tangent. However, consideration of the quadrant containing (x, y) will give a unique $\theta \in [0, 2\pi)$, provided $r > 0$ (we do not define θ when $r = 0$).

Addition of complex numbers is defined coordinatewise:

$$(a, b) + (c, d) = (a + c, b + d),$$

and can be visualized by vector addition. See Figure 1.2.

Multiplication is given by

$$(a, b) \cdot (c, d) = (ac - bd, bc + ad)$$

and can be visualized as follows. The points $(0, 0), (1, 0), (a, b)$ form a triangle. Construct a similar triangle with corresponding points $(0, 0)$, (c, d), (x, y). Then it is an exercise in high-

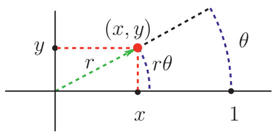

Figure 1.1 Cartesian and polar representations of complex numbers.

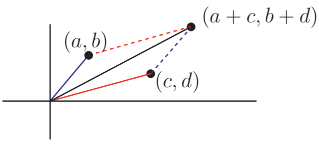

Figure 1.2 Addition.

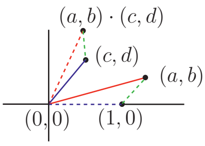

Figure 1.3 Multiplication.

school geometry to show that $(x, y) = (a, b) \cdot (c, d)$. By similarity, the length of the product is the product of the lengths and polar coordinate angles are added. See Figure 1.3.

The real number t is identified with the complex number $(t, 0)$. With this identification, complex addition and multiplication are extensions of the usual addition and multiplication of real numbers. For conciseness, when t is real, $t(x, y)$ means $(t, 0) \cdot (x, y) = (tx, ty)$. The additive identity is $0 = (0, 0)$ and $-(x, y) = (-x, -y)$. The multiplicative identity is $1 = (1, 0)$ and the multiplicative inverse of (x, y) is $(x/(x^2 + y^2), -y/(x^2 + y^2))$. It is a tedious exercise to check that the commutative and associative laws of addition and multiplication hold, as does the distributive law.

The notation for complex numbers becomes *much* easier if we use a single letter instead of a pair. It is traditional, at least among mathematicians, to use the letter i to denote the complex number $(0, 1)$. If z is the complex number given by (x, y), then, because $(x, y) = x(1, 0) + y(0, 1)$, we can write $z = x + yi$. If $z = x + iy$, then the **real part** of z is $\text{Re} z = x$ and the **imaginary part** is $\text{Im} z = y$. Note that $i \cdot i = -1$. We can now just use the usual algebraic rules for manipulating complex numbers together with the simplification $i^2 = -1$. For example, z/w means multiplication of z by the multiplicative inverse of w. To find the real and imaginary parts of the quotient, we use the analog of "rationalizing the denominator":

$$\frac{x + iy}{a + ib} = \frac{(x + iy)(a - ib)}{(a + ib)(a - ib)} = \frac{xa - i^2 yb + iya - ixb}{a^2 + b^2}$$
$$= \frac{xa + yb}{a^2 + b^2} + \frac{ya - xb}{a^2 + b^2} i.$$

Here is some additional notation: if $z = x + iy$ is given in polar coordinates by the pair (r, θ) then

$$|z| = r = \sqrt{x^2 + y^2}$$

is called the **modulus** or **absolute value** of z. Note that $|z|$ is the distance from the complex number z to the origin 0. The angle θ is called the **argument** of z and is written

$$\theta = \arg z.$$

The most common convention is that $-\pi < \arg z \leq \pi$, where positive angles are measured counter-clockwise and negative angles are measured clockwise. The complex conjugate of z is given by

$$\bar{z} = x - iy.$$

The complex conjugate is the reflection of z about the **real line** $\mathbb{R}$.

It is an easy exercise to show the following:

$$\begin{aligned}
|zw| &= |z||w|,\\
|cz| &= c|z| \text{ if } c > 0,\\
z/|z| &\text{ has absolute value } 1,\\
z\bar{z} &= |z|^2,\\
\text{Re}z &= (z + \bar{z})/2,\\
\text{Im}z &= (z - \bar{z})/(2i),\\
\overline{z + w} &= \bar{z} + \bar{w},\\
\overline{zw} &= \bar{z} \cdot \bar{w},\\
\bar{\bar{z}} &= z,\\
|z| &= |\bar{z}|,\\
\arg zw &= \arg z + \arg w \quad \text{modulo } 2\pi,\\
\arg \bar{z} &= -\arg z = 2\pi - \arg z \quad \text{modulo } 2\pi.
\end{aligned}$$

The statement **modulo** 2π means that the difference between the left- and right-hand sides of the equality is an integer multiple of 2π.

The identity $a + (z - a) = z$ expressed in vector form shows that $z - a$ is (a translate of) the vector from a to z. Thus $|z - a|$ is the length of the complex number $z - a$ but it is also equal to the distance from a to z. The circle centered at a with radius r is given by $\{z : |z - a| = r\}$ and the disk centered at a of radius r is given by $\{z : |z - a| < r\}$. The open disks are the basic open sets generating the standard topology on $\mathbb{C}$. We will use $\mathbb{D}$ to denote the **unit disk**,

$$\mathbb{D} = \{z : |z| < 1\},$$

and use $\partial\mathbb{D}$ to denote the **unit circle**,

$$\partial\mathbb{D} = \{z : |z| = 1\}.$$

Complex numbers were around for at least 250 years before good applications were found; Cardano discussed them in his book *Ars Magna* (1545). Beginning in the 1800s, and continuing today, there has been an explosive growth in their usage. Now complex numbers are very important in the application of mathematics to engineering and physics.

It is a historical fiction that solutions to quadratic equations forced us to take complex numbers seriously. How to solve $x^2 = mx + c$ has been known for 2000 years and can be visualized as the points of intersection of the standard parabola $y = x^2$ and the line $y = mx+c$. As the line is shifted up or down by changing c, it is easy to see there are two, one or no (real) solutions. The solution to the cubic equation is where complex numbers really became important. A cubic equation can be put in the standard form

$$x^3 = 3px + 2q$$

by scaling and translating. The solutions can be visualized as the intersection of the standard cubic $y = x^3$ and the line $y = 3px + 2q$. *Every* line meets the cubic, so there will always be a solution. By formal manipulations, Cardano showed that a solution is given by

$$x = (q + \sqrt{q^2 - p^3})^{\frac{1}{3}} + (q - \sqrt{q^2 - p^3})^{\frac{1}{3}}.$$

Bombelli pointed out 30 years later that if $p = 5$ and $q = 2$, then $x = 4$ is a solution, but $q^2 - p^3 < 0$ so the above solution does not make sense. His "wild thought" was to use complex numbers to understand the solution

$$x = (2 + 11i)^{\frac{1}{3}} + (2 - 11i)^{\frac{1}{3}}.$$

He found that $(2 \pm i)^3 = 2 \pm 11i$, and so the above solution actually equals 4. In other words, complex numbers were used to find a real solution. This is not just an oddity of Cardano's formula, because, for some cubics, complex numbers must be used in any rational formula involving radicals by a theorem of O. Hölder [15]. See Exercises 1.9 and 1.10 for solutions of cubic and quartic equations.

1.2 Estimates

Here are some elementary estimates which the reader should check:

$$-|z| \leq \text{Re}z \leq |z|,$$
$$-|z| \leq \text{Im}z \leq |z|$$

and

$$|z| \leq |\text{Re}z| + |\text{Im}z|.$$

Perhaps the most useful inequality in analysis is the triangle inequality.

Theorem 1.1 (triangle inequality)

$$|z + w| \leq |z| + |w|$$

and

$$|z + w| \geq \big||z| - |w|\big|.$$

The associated picture perhaps makes this result geometrically clear. See Figure 1.4. Analysis is used to give a more rigorous proof of the triangle inequality (and it is good practice with the notation we have introduced).

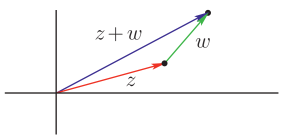

Figure 1.4 Triangle inequality.

Proof

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) \\
&= z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} \\
&= |z|^2 + 2\text{Re}(w\bar{z}) + |w|^2 \\
&\leq |z|^2 + 2|w||\bar{z}| + |w|^2 \\
&= (|z|+|w|)^2.
\end{aligned}$$

To obtain the second part of the triangle inequality we use

$$|z| = |z+w+(-w)| \leq |z+w| + |-w| = |z+w| + |w|.$$

By subtracting $|w|$,

$$|z| - |w| \leq |z+w|,$$

and switching z and w,

$$|w| - |z| \leq |z+w|,$$

so that

$$||z| - |w|| \leq |z+w|.$$ □

These estimates can be used to prove that $\{z_n\}$ converges if and only if both $\{\text{Re}z_n\}$ and $\{\text{Im}z_n\}$ converge. The series $\sum a_n$ is said to **converge** if the sequence of partial sums

$$S_m = \sum_{n=1}^{m} a_n$$

converges, and the series **converges absolutely** if $\sum |a_n|$ converges. A series is said to **diverge** if it does not converge. Absolute convergence implies convergence because Cauchy sequences converge. We sometimes write $\sum |a_n| < \infty$ to denote absolute convergence because the partial sums are increasing. It also follows that $\sum a_n$ is absolutely convergent if and only if both $\sum \text{Re}a_n$ and $\sum \text{Im}a_n$ are absolutely convergent. By comparing the nth partial sum and the $(n-1)$st partial sum, if $\sum a_n$ converges then $a_n \to 0$. The converse statement is false, for example if $a_n = 1/n$.

Another useful estimate is the Cauchy–Schwarz inequality.

Theorem 1.2 (Cauchy–Schwarz inequality)

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right| \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |b_j|^2\right)^{\frac{1}{2}}.$$

If v and w are vectors in $\mathbb{C}^n$, the Cauchy–Schwarz inequality says that $|\langle v, w\rangle| \leq ||v||||w||$, where the left-hand side is the absolute value of the inner product and the right-hand side is the product of the lengths of the vectors.

Proof The square of the right-hand side minus the square of the left-hand side in the Cauchy–Schwarz inequality can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(|a_j|^2|b_i|^2 - a_j\overline{b_j}\,\overline{a_i\overline{b_i}}\right).$$

We can add another copy of this quantity, switching the index i and the index j to obtain

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(|a_j|^2|b_i|^2 + |a_i|^2|b_j|^2 - a_j\overline{b_j}\,\overline{a_i}b_i - a_i\overline{b_i}\,\overline{a_j}b_j\right).$$

Using the identity $|A-B|^2 = |A|^2 + |B|^2 - A\overline{B} - \overline{A}B$, with $A = a_jb_i$ and $B = a_ib_j$, we obtain

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|a_jb_i - a_ib_j|^2. \qquad \square$$

The above proof also gives the error

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}|a_jb_i - a_ib_j|^2,$$

and so equality occurs if and only if $a_j = cb_j$ for all j and some (complex) constant c, or $b_j = 0$ for all j.

The reader can use Riemann integration to deduce the following, which is also called the Cauchy–Schwarz inequality. For a complex-valued function f defined on a real interval $[a,b]$, we define $\int_a^b f\,dx \equiv \int_a^b \mathrm{Re}f\,dx + i\int_a^b \mathrm{Im}f\,dx$.

Corollary 1.3 *If f and g are continuous complex-valued functions defined on $[a,b] \subset \mathbb{R}$ then*

$$\left|\int_a^b f(t)\overline{g(t)}dt\right| \le \left(\int_a^b |f(t)|^2dt\right)^{\frac{1}{2}}\left(\int_a^b |g(t)|^2dt\right)^{\frac{1}{2}}.$$

This corollary can also be proved directly by expanding

$$\int_a^b\int_a^b |f(x)g(y) - f(y)g(x)|^2dxdy \tag{1.1}$$

in a similar way, giving a proof for square integrable functions f, g. Moreover, the error term is half of the integral (1.1) and equality occurs if and only if $f = cg$, for some constant c, or g is identically zero.

1.3 Stereographic Projection

A component of Riemann's point of view of functions as mappings is that ∞ is like any other complex number. But we cannot extend the definition of complex numbers to include ∞ and still have the usual laws of arithmetic hold. However, there is another "picture" of complex

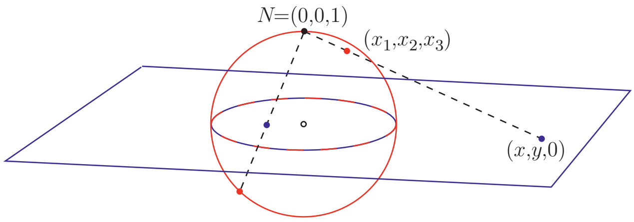

Figure 1.5 Stereographic projection.

numbers that can help us visualize this idea. The picture is called **stereographic projection**. We identify the complex numbers with the plane $\{(x,y,0) : x,y \in \mathbb{R}\}$ in $\mathbb{R}^3$. If $z = x + iy$, let z^* be the unique point on the unit sphere in $\mathbb{R}^3$ which also lies on the line from the **north pole** $(0,0,1)$ to $(x,y,0)$. Thus

$$z^* = (x_1, x_2, x_3) = (0,0,1) + t[(x,y,0) - (0,0,1)].$$

See Figure 1.5.

Then

$$|z^*| = \sqrt{(tx)^2 + (ty)^2 + (1-t)^2} = 1,$$

which gives

$$t = \frac{2}{x^2 + y^2 + 1},$$

where $0 < t \le 2$, and

$$z^* = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).$$

The reader is invited to find $z = x + iy$ from $z^* = (x_1, x_2, x_3)$. The sphere is sometimes called the **Riemann sphere** and is denoted $\mathbb{S}^2$. We can extend stereographic projection $\pi : \mathbb{C} \to \mathbb{S}^2$ to the **extended plane** $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ by defining $\pi(\infty) = (0,0,1)$. We give $\mathbb{C}^*$ the topology inherited from $\mathbb{S}^2$. It is an explicit one-point compactification of the complex plane. The north pole corresponds to "∞."

Theorem 1.4 *Under stereographic projection, circles and straight lines in $\mathbb{C}$ correspond precisely to circles on $\mathbb{S}^2$.*

Proof Every circle on the sphere is given by the intersection of a plane with the sphere, and conversely the intersection of a plane with a sphere is a circle or a point. See Exercise 1.6. If a plane is given by

$$Ax_1 + Bx_2 + Cx_3 = D,$$

and if (x_1, x_2, x_3) corresponds to $(x, y, 0)$ under stereographic projection, then

$$A\left(\frac{2x}{x^2+y^2+1}\right) + B\left(\frac{2y}{x^2+y^2+1}\right) + C\left(\frac{x^2+y^2-1}{x^2+y^2+1}\right) = D. \tag{1.2}$$

Equivalently,

$$(C - D)(x^2 + y^2) + 2Ax + 2By = C + D. \tag{1.3}$$

If $C = D$, then this is the equation of a line, and all lines can be written this way. If $C \neq D$, then, by completing the square, we get the equation of a circle, and all circles can be put in this form. □

So we will consider a line in $\mathbb{C}$ as just a special kind of **"circle."**

The sphere $\mathbb{S}^2$ inherits a topology from the usual topology on $\mathbb{R}^3$ generated by the balls in $\mathbb{R}^3$.

Corollary 1.5 *The topology on $\mathbb{S}^2$ induces the standard topology on $\mathbb{C}$ via stereographic projection, and moreover a basic neighborhood of ∞ is of the form $\{z : |z| > r\}$.*

For later use, we note that the chordal distance between two points on the sphere induces a metric, called the **chordal metric**, on $\mathbb{C}$ which is given by

$$\chi(z, w) = |z^* - w^*| = \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}. \tag{1.4}$$

This metric is bounded (by 2). See Exercise 1.5.

1.4 Exercises

A

1.1 Check that item 9 of the prerequisites holds for complex $a_{n,k}$. Check that items 13, 14 and 15 of the prerequisites hold for complex-valued functions defined on an interval $I \subset \mathbb{R}$.

1.2 Check the details of the high-school geometry problem in the geometric version of complex multiplication.

1.3 Prove the parallelogram equality:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

In geometric terms, the equality says that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the sides. It is perhaps a bit easier to prove it using the complex notation of this chapter than to prove it using high-school geometry.

1.4 Prove Corollary 1.3.

1.5 (a) Prove formula (1.4). An algebraic proof can be found in [1], p. 20. Alternatively, use the law of cosines for the triangles with vertices $N = (0, 0, 1)$, z, w and N, $z*$, $w*$. Compute edge lengths of these two triangles using triangles that have N and $(0, 0, 0)$ as vertices.

(b) The chordal distance is bounded by 2, by the triangle inequality. Verify analytically that the formula for this distance given in the text is bounded by 2 using the

Cauchy–Schwarz inequality, and also by directly multiplying it out using complex notation and one of the estimates at the start of Section 1.2.

1.6 Show that the intersection of a plane with the unit sphere in $\mathbb{R}^3$ is a circle or a point and conversely that every circle or point on the sphere is equal to the intersection of the sphere with a plane. Hint: Rotate the plane and sphere so that the plane is parallel to the $(x, y, 0)$ plane.

1.7 (a) Suppose w is a non-zero complex number. Choose z so that $|z| = |w|^{\frac{1}{2}}$ and $\arg z = \frac{1}{2}\arg w$ or $\arg z = \frac{1}{2}\arg w + \pi$. Show that $z^2 = w$ in both cases, and that these are the only solutions to $z^2 = w$.

(b) The quadratic formula gives two solutions to the equation $az^2 + bz + c = 0$, when a, b, c are complex numbers with $a \neq 0$ because completing the square is a purely algebraic manipulation of symbols, and there are two complex square roots of every non-zero complex number by part (a). Check the details.

(c) If w is a non-zero complex number, find n solutions to $z^n = w$ using polar coordinates.

B

1.8 Suppose that f is a continuous complex-valued function on a real interval $[a, b]$. Let

$$A = \frac{1}{b-a}\int_a^b f(x)dx$$

be the average of f over the interval $[a, b]$.

(a) Show that if $|f(x)| \le |A|$ for all $x \in [a, b]$, then $f = A$. Hint: Rotate f so that $A > 0$. Then $\int_a^b (A - \text{Re}f)dx/(b-a) = 0$, and $A - \text{Re}f$ is continuous and non-negative.

(b) Show that if $|A| = (1/(b-a))\int_a^b |f(x)|dx$, then $\arg f$ is constant modulo 2π on $\{z : f(z) \neq 0\}$.

1.9 Formally solve the cubic equation $ax^3+bx^2+cx+d = 0$, where $x, a, b, c, d \in \mathbb{C}$, $a \neq 0$, by the following reduction process:

(a) Set $x = u + t$ and choose the constant t so that the coefficient of u^2 is equal to zero.

(b) If the coefficient of u is also zero, then take a cube root to solve. If the coefficient of u is non-zero, set $u = kv$ and choose the constant k so that $v^3 = 3v + r$, for some constant r.

(c) Set $v = z+1/z$ and obtain a quadratic equation for z^3. The map $z+1/z$ is important for several reasons, including constructing what are called conformal maps. It will be examined in more detail in Section 6.4.

(d) Use the quadratic formula to find two possible values for z^3, and then take a cube root to solve for z.

(e) In Section 2.2 we will show that the cubic equation has exactly three solutions, counting multiplicity. But the process in this exercise appears to generate more solutions, if we use two solutions to the quadratic and all three cube roots. Moreover, there might be more than one valid choice for the constants used to reduce to a simpler equation. Explain.

1.10 The equation

$$a\left(z+\frac{1}{z}\right)^2+b\left(z+\frac{1}{z}\right)+c=0$$

has four solutions, which can be found by two applications of the quadratic formula. If we multiply by z^2 we obtain the quartic

$$az^4+bz^3+(2a+c)z^2+bz+a=0.$$

Which quartics $Aw^4+Bw^3+Cw^2+Dw+E$ can be put in this form after a linear change of variable $w=\alpha z+\beta$?

C

1.11 Prove that stereographic projection preserves angles between curves. In other words, if two curves γ_1, γ_2 in the plane meet at an angle θ, then their lifts γ_1^*, γ_2^* to the sphere meet at the same angle. Moreover, the direction of the angle from γ_1 to γ_2 corresponds to the direction from γ_1^* to γ_2^* when viewed from inside the sphere. Orientation is reversed when viewed from outside the sphere. Hint: This can be done without any calculations by considering intersecting planes. This exercise will be revisited in Exercise 6.16, where it is used to find the Mercator projection, a map of tremendous economic impact.

1.12 Stereographic projection combined with rigid motions of the sphere can be used to describe some transformations of the plane.

(a) Map a point $z\in\mathbb{C}$ to $\mathbb{S}^2$, apply a rotation of the unit sphere, then map the resulting point back to the plane. For a fixed rotation, find this map of the extended plane to itself as an explicit function of z. Two cases are worth working out first: rotation about the x_3 axis and rotation about the x_1 axis.

(b) Another map can be obtained by mapping a point $z\in\mathbb{C}$ to $\mathbb{S}^2$, then translating the sphere so that the origin is sent to (x_0, y_0, z_0), then projecting back to the plane. The projection to the plane is given by drawing a line through the (translated) north pole and a point on the (translated) sphere and finding the intersection with the plane $\{(x, y, 0)\}$. For a fixed translation, find this map as an explicit function of z. In this case it is worth working out a vertical translation and a translation in the plane separately. Then view an arbitrary translation as a composition of these two maps. Partial answer: the maps in parts (a) and (b) are of the form $(az+b)/(cz+d)$ with $ad-bc\neq 0$.
For an award-winning movie of these maps, see
http://www-users.math.umn.edu/~arnold/moebius/
but do the exercise before viewing this link.