Introduction

Responding to widespread interest within cultural studies and social inquiry, this book takes up the question of what a mathematical concept is, using a variety of vanguard theories in the humanities and posthumanities. Tapping into historical, philosophical, mathematical, sociological and psychological perspectives, each chapter explores the question of how mathematics comes to matter. Of interest to scholars across the usual disciplinary divides, this book tracks mathematics as a cultural and material activity. Unlike other books in this area, this book is highly interdisciplinary, devoted to exploring the ontology of mathematics as it plays out in empirical contexts, offering readers a diverse set of crisp and concise chapters.

The framing of the titular question is meant to be simple and direct, but each chapter unpacks this question in various ways, modifying or altering it as need be. Authors develop such variations as:

1. When does a mathematical concept become a mathematical concept?
2. What is the relationship between mathematical concepts, discourse and the material world?
3. How might alternative ontologies of mathematics be at work at this historical moment?
4. How do our theories of cognition and learning convey particular assumptions about the nature of mathematical concepts?
5. How might we theorize processes of mathematical abstraction and formalisation?
6. What is the role of diagrams, symbols and gestures in making mathematical concepts?
7. How do mathematical concepts inform particular ideological positions?

The authors take up these questions using tools from philosophy, anthropology, sociology, history, discursive psychology and other fields, provoking readers to interrogate their assumptions about the nature of mathematical concepts. Thus, the book presents a balance of chapters, diverse in their application but unified in their aim of exploring the central question. Each chapter examines in some detail case studies and examples, be they historical or situated in contemporary practice and public life. Each author explores the historical and situated ways that mathematical concepts come to be valued. Such focus allows for a powerful investigation into how mathematical concepts operate on various material planes, making the book an important contribution to recent debates about the nature of mathematics, cognition and learning theory. In offering a set of diverse and operational approaches to rethinking the nature of mathematics, we hope that this book will have far-reaching impact across the social sciences and the humanities. Authors delve into particular mathematical habits – creative diagramming, tracking invariants, structural mappings, material agency, interdisciplinary coverings – in order to explore the many different ways that mathematical concepts come to populate our world.

THE CONTEXT FOR THIS BOOK: PHILOSOPHY AND COGNITION

This book springs from our desire to pursue a cultural studies of mathematics that incorporates philosophy, history, sociology, and learning theory. We conceived this book as a collection of essays exploring and in some sense reclaiming a canonical question – what is a mathematical concept? – from the philosophy of mathematics. Authors take up this question innovatively, tapping into new theory to examine contemporary mathematics and current contexts. For those unfamiliar with the philosophy of mathematics, this section briefly recounts how this canonical question was typically addressed in the past. The ontology of concepts has long been a central concern for philosophers, and many of these philosophers considered the mathematical concept as an exemplary case for their investigations. The conventional starting point has tended to be framed as a dichotomy: Do mathematical concepts exist inside or outside the mind? From this starting point, further binaries are encountered: If concepts exist outside the mind, are they corporeal or incorporeal? If they are corporeal, do they exist in the things that are perceptible by the senses or are they separate (or independent) from them? Bostock (2009) suggests that philosophers have typically taken three positions in relation to such questions: cognitive, realist and nominalist.[1] These conventional responses have dominated the philosophy

[1] We have changed Bostock's term "conceptualist" to "cognitive" better to name its focus on *mental* concepts, and to avoid any confusion with how the term is used in our book.

of mathematics in previous centuries, and have become somewhat ossified in their characterization. This book charts entirely new territory, and yet for the sake of context it is worth describing very briefly these three schools of thought, and tracing their influence on twentieth-century constructivist theories of learning. This will set the stage for the post-constructivist approaches that are used in this book.

The cognitive approach claims that concepts exist in the mind and are created by the mind. Descartes, Locke and Kant, to some degree, might be considered to be in this camp. According to some variants of the cognitive approach, humans create universal, matter-independent concepts based on sense perception, while other variants claim that concepts are innate and do not require perceptual experience. In either case, concepts are treated as mental images or language-like entities. The second group of Bostock's philosophers, the realists (e.g., Plato, Frege and Gödel), claim that mathematical concepts exist outside the mind and are independent of all human thought, while the third group, the nominalists, claim that they do not exist at all, and are simply symbols or fictions.

Of course such sorting of philosophers into simplistic positions ignores the complexity of their thought, but it might help some readers, who are unfamiliar with the philosophy of mathematics, appreciate the radically divergent approaches developed in this book. Moreover, it is important to note how particular ideas from this tradition – such as Kant's theory that mathematical statements are "synthetic *a priori*" – have saturated many later developments in the philosophy of mathematics, seeping into the realist and nominalist camps as well. Brown (2008) indicates that Frege embraced Kant's view on geometry, Hilbert embraced Kant's view on arithmetic and even Russell can be characterized as Kantian in some crucial respects.

One might also argue that Kant's theory of mathematical truth has saturated theories of learning and has become full fledged in cognitive psychology and its dominant image of learning as that which entails acquiring a set of cognitive 'schemas'. Constructivist theories of learning, in which concepts are constructed rather than acquired, also tend to frame the constructed concept as a mental image. According to this approach, student capacity for developing mathematical concepts is based in part on inductively generalising from engagements with material objects and discourse. A constructivist approach to concept formation tends to centre on the epistemic subject who synthesizes and subsumes these diverse materials and social encounters under one *cognitive* concept. Accordingly, concepts are treated as abstractions that ultimately transcend the messy world of hands, eyes, matter and others.

Constructivist theories of concept formation find their usual source in the work of either Piaget or Vygotsky. In the former case, Piaget's notion of *reflective abstraction* has been used to describe what it means to learn or develop a concept. Piaget spoke of four different types of abstractions, but the notion of reflective abstraction that was adopted by many education researchers involves the dual process of projection (borrowing existing knowledge from a preceding level of thought to use at a higher level) and conscious reorganization of thought into a new structure (becoming aware of what has been abstracted in that projection). For Piaget, reflective abstraction was the mechanism through which all mathematical structures were constructed. In his genetic epistemology approach, he broke with existing theories of concept development found both in philosophy and psychology because he based his analyses on empirical observations of children's activity. For example, in the case of number, Piaget combined the relational and classificatory concepts of number, which had been seen as incommensurable by philosophers at the time (Brainerd, 1979). This focus on the mathematical activity of non-experts introduced important insights that philosophers had overlooked. On the other hand, researchers today who follow in the Piagetian tradition (see, for example, Simon et al., 2016) tend to pay little attention to philosophical considerations of particular mathematical concepts, focusing exclusively on the trajectories of particular children working on particular tasks.

For Vygotsky, concept formation was goal-oriented and entirely social: "A concept emerges and takes shape in the course of a complex operation aimed at the solution of some problem" (1934, p. 54); "A concept is not an isolated, ossified, and changeless formation" (Vygotsky, p. 98). Vygotsky saw concept formation as necessarily being mediated by signs (principally language and material tools); for instance, he argued that language is the means by which a learner focuses attention and makes distinctions within the environment, distinctions that can be analysed and synthesized. As with Piaget, Vygotsky insisted that concepts could not be taught directly, and that concept formation was a long and complex process. Whereas spontaneous concepts could be developed from direct experience of the world through induction, scientific concepts develop through deduction and require exposure (through school, for example) to abstract cultural knowledge and different forms of reasoning. Thus, one way of characterizing the difference between Piaget and Vygotsky is that for the former, reflective abstractions begin with the actions of the individual and are then shared out in the social realm, while for the latter, scientific concepts begin in the social realm and are internalized by the individual. Researchers working through a Vygotskian perspective today focus strongly on the role that language and tools play in learners' concept formation, as well as on the teacher actions that support the process of internalization (see, for example, Mariotti, 2013).

The tendency for researchers influenced by both Piaget and Vygotsky to focus almost exclusively on the psychological nature of concepts may account for DiSessa and Sherin's (1998) critique of current educational work on concepts. In their attempt to formalise "conceptual change", they note that one of the main difficulties in most accounts is "the failure to unpack what 'the very concepts' are in sufficiently rigorous terms" (p. 1158). This frustration might stem in part from the fact that researchers cannot *see* the schemes or structures that are posited by Piaget's account of reflective abstraction, or even the process of internalisation described by Vygotsky.

In the context of education research, concepts are often distinguished from memorized facts and procedures, and often qualified in terms of misconceptions and protoconceptions. Curriculum policy advocates for the importance of conceptual understanding, and typically stipulates which mathematical concepts are most important in teaching and learning. But this kind of listing of key concepts offers little insight into the specific nature of mathematical concepts and the material-historical processes associated with them.

Recent developments in post-constructivist learning theories have shown how concepts are performed, enacted or produced in gestures and other material activities (Davis, 2008; Hall & Nemirovsky, 2011; Radford, 2003; Roth, 2010). This new theoretical shift draws attention to how concepts are formed in the activity itself rather than in the rational cognitive act of synthesizing (Brown, 2011; Tall, 2011). This work reflects a paradigmatic shift in learning theory, driven in large part by offshoots of contemporary phenomenology, better to address the role of the body in coming to know mathematics.

There are yet further developments on this front, developments that build on the phenomenological tradition, and diverge from it in significant ways. For instance, Deleuze and Guattari (1994), whose work is cited often in this book, reanimate the concept as part of their philosophy of immanence. They propose a "pedagogy of the concept", by which concepts are to be treated as creative devices for carving up matter, rather than pure forms subject only to recognition. This pedagogy of the concept aims to encounter and engage with the conceptual on the material plane; a concept brings with it an entire "plane of immanence" (Cutler & MacKenzie, 2011, p. 64). For Stengers (2005), Deleuze's pedagogy is about learning "the 'taste' of concepts, being modified by the encounter with concepts" (p. 162). de Freitas and Sinclair (2014) have developed this post-humanist approach to concept formation, arguing that learning is about encountering the mobility and indeterminacy of concepts.

This book takes up these recent developments to explore new ontologies of mathematics and pushes against all-too-easy dualisms between matter and meaning. It does so by taking a broad view of concepts to include their historical and cultural dimensions, their trajectories in and through classrooms and their potentially changing nature within contemporary mathematics. The chapters dig deep into mathematical practice and culture, troubling conventional approaches and their constructivist offspring.

Our hope is that this book contributes to the philosophy of mathematics (how does mathematics evolve as a discipline? How are concepts formed and shared?), as well as cultural studies of mathematics (How do mathematical concepts format worldviews? How do they participate in the creation of political and social discourse?). We also hope that the book triggers discussions about significant questions within mathematics education, such as: How might learning theories change if we view concepts as generative of new spacetime configurations rather than timeless, determinate and immovable? What happens to curriculum when we treat concepts as material assemblages, temporally evolving and vibrating with potentiality?

THEMES AND CHAPTERS

The first two chapters are by Michael J. Barany and Reviel Netz, respectively, who each provide some more historical context (and critique) of theories of mathematical concept construction. Barany engages in some long-standing considerations of the epistemological status of mathematical concepts, with a particular interest in the principle of *meaning finitism*, which emerged from sociology of scientific knowledge (SSK) perspectives that gained currency in the 1970s. This perspective stresses the contingent human aspects of mathematical knowledge, particularly through the activities of labelling and classifying. Barany uses Lakatos' account of the development of the concept of polyhedron to exemplify a "meaning finitism" account of mathematics. Rather than focus on more ontological debates about the status of simple objects (numbers, shapes), Barany focuses on how mathematical concepts are used and revised over time.

Netz's chapter raises the question of what it means for mathematics to be conceptual, especially in the context of historical situations. He describes many claims that have been made about whether or not certain cultures possessed a particular mathematical concept. He highlights two ways in which such claims might be misleading. The first relates to what we might call frequency of use. Netz shows several examples of a concept existing in a certain culture without it becoming widespread or frequently used. The second, perhaps more interesting to mathematicians, relates to conceptual hierarchy. By showing persuasively how Archimedes used the concept of actual infinity, Netz troubles common assumptions that the concept of actual infinity depends on the concept of set. As Barany's meaning finitism would make evident, the particular ways in which knowledge is classified (ordered, related) is highly contingent and cannot be assumed to play out in the same way in different historical periods and different geographical locations. Indeed, Netz highlights how different mathematical *practices* give rise to different concepts.

The next two chapters continue to look at the material practices of mathematical activity, exploring how mathematical concepts live through various media. Juliette Kennedy examines the role of visualization and diagramming in mathematics, and asks whether some mathematical concepts are irreducibly visual. She focuses on the role of these informal "co-exact" characteristics of mathematical drawing for the part they play in logical inference, first tracking the historical separation of the visual from the logical. The chapter by Elizabeth de Freitas and Nathalie Sinclair attends to the historical division between logic and mathematics in a related way, looking at the concept of the mathematical continuum, to show that number and line are mathematical concepts which are the source of persistent philosophical questions about space, time and mobility. Just as Kennedy talks about the "bidirectionality" of mathematical practice (between body and symbol) and the "ambivalence" entailed in mathematical positioning, de Freitas and Sinclair suggest that mathematical concepts are always rumbling beneath the apparent foundations of mathematical truth. They draw on the ideas of Gilles Châtelet and Ian Hacking to show how concepts thrive through material media and historical material arrangements. These two chapters challenge readers to reconsider the way that proof and reasoning is at play in mathematics.

Kennedy first distinguishes between drawings that are directly constitutive of a mathematical proof and others that are informal, "incidental" aspects of mathematical activity, discussing how both kinds function fruitfully in mathematics. She discusses "world-involving inference" and logical inference, seeking a middle synthetic ground where mixtures of reasoning operate. Drawing on the reflections of the architect Juhani Pallasmaa about "the thinking hand", Kennedy argues that the manual activity of mathematical drawing must be considered as we ask the question: What is a mathematical concept? Mathematicians move around a mathematical diagram much like one might move around a building, and it is through this habitation and spatial practice that the concepts become known. This chapter also links to that by Nemirovsky, who describes how one comes to inhabit a concept over time, through habitual carving out of its contour and meaning.

The chapter by de Freitas and Sinclair continues the theme that Kennedy opens, regarding the relationship between the logical and the mathematical. They cite Hacking (2014), who argues that the connection between symbolic logic and mathematics "simply did not exist" until the logicist movement of the nineteenth century (advocated by Frege, above all), which aimed to reduce mathematics to logic, and replaced Aristotelian logic with what was termed "symbolic logic" (p. 137). This chapter proposes the term "virtual" to describe the indeterminate dimension in matter that literally destabilizes the rigidity of extension. They suggest that concepts such as line, point and circle can be conceived using a genetic definition that emphasizes the dynamic and mobile aspects of mathematical concepts. Concepts – such as squareness, fiveness, etc. – thus retain the trace of the movement of the eye, hand and thinking body. This chapter is linked to the one by Netz, as they both present images of mathematical practice as an applied or practical affair, grounded in material conditions and experiments rather than exclusive appeals to logic.

Chapters by Arkady Plotnitsky and Simon Duffy explore the ways in which mathematical concepts spring from and sustain rich problem spaces. They both draw on the powerful ideas of Gilles Deleuze and Felix Guattari to develop a theory of mathematical concepts, and then show its relevance to other discourses. Deleuze, in particular, offered deep insights into the history of mathematics, tapping particular ideas – from Galois, Riemann, Poincaré, Lautman and others – to rethink the relationship between concepts and problems. We see in Plotnitsky and Duffy's chapters a theoretical move that explores the speculative position of a "*mathesis universalis*" (Deleuze, 1994, p. 181), but not one that posits a definite system of mathematical laws at the base of nature. Rather, these two chapters delve into the mathematical concept as that which operates through a rich dynamic ontology of problems that are in some way shared with other discourses and contexts.

Plotnitsky explores the contributions of Bernhardt Riemann around non-Euclidean geometry, also drawing on the insights of Deleuze. Riemann's work is known as a conceptual rather than axiomatic approach to exploring non-Euclidean geometries. Plotnitsky uses the work of Riemann to show that a mathematical concept (1) emerges from the co-operative confrontation between mathematical thought and chaos; (2) is multi-component; (3) is related to or is a problem; and (4) has a history. Plotnitsky argues that mathematical concepts are not simply referents or functional objects, but that they tap into a "plane of immanence", which is a Deleuzian term that describes the vibrant virtual realm of potentiality *in* the world. The plane of immanence is the plane of the movement of *philosophical* thought that gives rise to *philosophical* concepts, but Plotnitsky argues that mathematics also creatively operates through this plane of immanence. In particular, Plotnitsky shows how mathematical concerns regarding the distinction between discrete and continuous manifolds are philosophical in the Deleuzian sense. Thus, Plotnitsky shows that mathematics as much as philosophy engages with "chaos" by creating planes of immanence and concepts. He argues that creative *exact* mathematical and scientific thought is defined by planes of immanence and invention of *exact* concepts, the architecture of which is analogous to that of philosophical concepts in Deleuze and Guattari's sense.

Duffy shows how a *practice of mathematical problems* – using the examples of the problem of solving the quintic and the problem of the diagrammatic representation of essential singularities – operates as the engine of mathematical invention, such that the emergent "solutions" are clusters of concepts that carry with them the problem space from which they emerged. In other words, following Lautman, concepts are inherently problematic and carry with them the force of the problem – indeed, this force animates them. Duffy shows how Deleuze is ultimately interested in how this theory of mathematical problems offers even broader significance because it can be deployed *as a way of studying problems and concepts in other discourses, or fields and contexts*. In particular, Duffy shows how Deleuze's work in his seminal *Difference and repetition* (1994) deploys the conceptual space of the early mathematical calculus to rethink the nature of perception. It is not, however, that Deleuze privileges the discourse of mathematics over others in some absolute sense, but rather that it offers distinctive insights (just as any other might) into our shared ontology.

The chapters by David Corfield and Michael Harris both consider the emergence of new concepts in mathematics, in a contemporary setting. Corfield's chapter is concerned with homotopy type theory while Harris traces the recent emergence of the perfectoid. Corfield's interest in homotopy type theory stems from the way it exemplifies the vertical unity of mathematics. For Harris, the focus is on how the concept of the perfectoid came to be seen as "the right" concept within the mathematics community – a story he offers as a participant-observer. Both authors highlight how mathematical concepts are tied up in axiological concerns. While

Harris refuses to offer criteria for what makes a concept "good", he draws attention to the many social and historical factors – such as the connection to Grothendieck and perhaps even the endearing personality of Scholze – that converged to make the perfectoid the 'right' concept for solving a set of diverse mathematical problems. He chronicles the way in which the perfectoid concept was put to work extensively by Scholze and others, almost like a kind of mutant offspring of current theories. This suggests that the applicability of a concept (where the application is across mathematics, rather than outside of mathematics), is a highly generative process whereby new practices emerge that change the entire field.

Similarly, Corfield provides a compelling argument for the "goodness" of homotopy type theory, which has developed a strong footing in the past decade. Corfield describes how this theory, and type theory more generally, exploits the vertical unity of mathematics. Such unity entails consistency demands, but perhaps also points to uncharted pedagogical terrain. There are some important nuances to keep in mind, which Corfield highlights in his discussion of Mark Wilson's insistence on the "wandering" nature of concepts and his warning that "hazy holism" can often misleadingly lead us to believe in the unity of concepts, which are more often than not "patched together from varied parts" (p. 129). The very practice of *patching* becomes pivotal to Corfield's considerations of the 'spatial' nature of homotopy type theory.

Thus we might also see the vertical unity as arising from a patching together of different kinds of mathematical practices, much as we saw in Harris' chapter. That strong analogies can be seen across basic arithmetic and homotopy is convincingly and carefully shown by Corfield, but one look at the syntactic complexity required to "express" addition or inverse in homotopy type theory is enough to remind us that these are not the same concepts. We are reminded of Thurston's (1994) description of the different ways of thinking about the derivative. While the differences may "start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions" (p. 3), they are much more real in the particular contexts in which they are actually used. Staying close to particular practices – rather than erasing those differences within a reductive set theory – allows Corfield to seek out other important "unities" across other concepts, such as formal and concrete duality.

The notion of vertical unity seems to us an interesting one for mathematics education, for how it troubles conventions about developmental conceptual change and curriculum. School mathematics has long been considered an edifice whose stairway must be climbed one step at a time. Vertical unity brings about some different imagery: express elevators, the possibility of starting at the penthouse of homotopy type theory, a