# 1

# Basic notions and steady flows

In this chapter, we define the subject, derive the equations of motion and describe their fundamental symmetries. We start from hydrostatics where all forces are normal. We then try to consider flows this way as well, neglecting friction. This allows us to understand some features of inertia, most importantly induced mass, but the overall result is a failure to describe a fluid flow past a body. We are then forced to introduce friction and learn how it interacts with inertia, producing real flows. We briefly consider an Aristotelean world where friction dominates. In an opposite limit, we discover that the world with a little friction is very much different from the world with no friction at all.

## 1.1 Definitions and basic equations

Here we define the notions of fluids and their continuous motion. These definitions are induced by empirically established facts rather than deduced from a set of axioms.

### 1.1.1 Definitions

We deal with *continuous media* where matter may be treated as homogeneous in structure down to the smallest portions. The term *fluid* embraces both liquids and gases and relates to the fact that even though any fluid may resist deformations, that resistance cannot prevent deformation from happening. This is because the resisting force vanishes with the rate of deformation. With patience, anything can be deformed. Therefore, whether one treats the matter as a fluid or a solid depends on the time available for observation. As the prophetess Deborah sang, “The mountains flowed before the Lord”

(Judges 5:5). The ratio of the relaxation time to the observation time is called the Deborah number.[1] The smaller the number the more fluid the material.

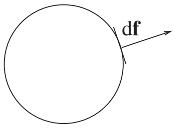
A fluid can be in mechanical equilibrium only if all the mutual forces between two adjacent parts are normal to the common surface. That *experimental* observation is the basis of hydrostatics. If one applies a force parallel (tangential) to the common surface then the fluid layer on one side of the surface starts sliding over the layer on the other side. Such sliding motion will lead to a friction between layers. For example, if you cease to stir tea in a glass it could come to rest only because of such tangential forces, i.e. friction. Indeed, if the mutual action between the portions on the same radius was wholly normal, i.e. radial, then the conservation of angular momentum about the rotation axis would cause the fluid to rotate forever.

Since tangential forces are absent at rest or for a uniform flow, it is natural to consider first the flows where such forces are small and can be neglected. Therefore, a natural first step out of hydrostatics into hydrodynamics is to restrict ourselves to purely normal forces, assuming small velocity gradients (whether such a step makes sense at all and how long such approximation may last remains to be seen). Moreover, the intensity of a normal force per unit area does not depend on the direction in a fluid (Pascal's law, see Exercise 1.1). We thus characterize the internal force (or stress) in a fluid by a single scalar function $p(\mathbf{r},t)$ called pressure, which is the force per unit area. From the viewpoint of the internal state of the matter, pressure is a macroscopic (thermodynamic) variable. Microscopically, we assume every portion of the fluid to be in thermal equilibrium. In this case, the internal state of the fluid is described completely by two variables, so one needs a second thermodynamical quantity. We shall usually use the density $\rho(\mathbf{r},t)$, in addition to the pressure.

What *analytic properties* of the velocity field $\mathbf{v}(\mathbf{r},t)$ do we need to presume? We suppose the velocity to be finite and a continuous function of $\mathbf{r}$. In addition, we suppose the first spatial derivatives to be everywhere finite. That makes the *motion continuous*, i.e. trajectories of the fluid particles do not cross. The equation for the distance $\delta\mathbf{r}$ between two close fluid particles is $\mathrm{d}\delta\mathbf{r}/\mathrm{d}t = \delta\mathbf{v}$ so, mathematically speaking, the finiteness of $\nabla\mathbf{v}$ is the Lipschitz condition for this equation to have a unique solution (a simple example of nonunique solutions for non-Lipschitz equation is $\mathrm{d}x/\mathrm{d}t = |x|^{1-\alpha}$ with *two* solutions, $x(t) = (\alpha t)^{1/\alpha}$ and $x(t) = 0$, starting from zero for $\alpha > 0$). For a continuous motion, any surface moving with the fluid completely separates matter on the two sides of it. We don't yet know when exactly the continuity assumption is consistent with the equations of the fluid motion. Whether velocity derivatives may turn into infinity after a finite time is a subject of active research for an incompressible viscous fluid (and a subject of a one-million-dollar Clay prize). We shall see that a compressible inviscid flow generally develops discontinuities, called shocks.

### 1.1.2 Equations of motion for an ideal fluid

**The Euler equation.** The force acting on any fluid volume is equal to the pressure integral over the surface: $-\oint p\,d\mathbf{f}$. The surface area element $d\mathbf{f}$ is a vector directed as outward normal:




Let us transform the surface integral into the volume one: $-\oint p\,d\mathbf{f} = -\int \nabla p\,dV$. The force acting on a unit volume is thus $-\nabla p$. That would be wrong, however, to assume that this force is the time derivative of the momentum $\rho\mathbf{v}$ of this volume. To write the second law of Newton, we need to single out a fixed body of fluid. An infinitesimal such body is called *fluid particle* and it always contains the same mass, which we assume unity. Then the force per unit mass, $\nabla p/\rho$, must be equal to the acceleration $d\mathbf{v}/dt$:

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla p}{\rho} .$$

The acceleration $d\mathbf{v}/dt$ is not the rate of change of the fluid velocity at a fixed point in space but the rate of change of the velocity of a given fluid particle as it moves about in space. One uses the chain rule of differentiation to express this (substantial or material) derivative in terms of quantities referring to points fixed in space. During the time $dt$ the fluid particle changes its velocity by $d\mathbf{v}$ (which is composed of two parts, temporal and spatial):

$$d\mathbf{v} = dt\frac{\partial \mathbf{v}}{\partial t} + (d\mathbf{r}\cdot\nabla)\mathbf{v} = dt\frac{\partial \mathbf{v}}{\partial t} + dx\frac{\partial \mathbf{v}}{\partial x} + dy\frac{\partial \mathbf{v}}{\partial y} + dz\frac{\partial \mathbf{v}}{\partial z} . \tag{1.1}$$

It is the change in the fixed point plus the difference at two points $d\mathbf{r}$ apart, where $d\mathbf{r} = \mathbf{v}dt$ is the distance moved by the fluid particle during $dt$ due to inertia. Dividing (1.1) by $dt$ we obtain the substantial derivative as a local derivative plus a convective derivative:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} .$$

We see that even when the flow is steady, $\partial\mathbf{v}/\partial t = 0$, the acceleration is nonzero as long as $(\mathbf{v}\cdot\nabla)\mathbf{v} \neq 0$, that is if the velocity field changes in space along itself.

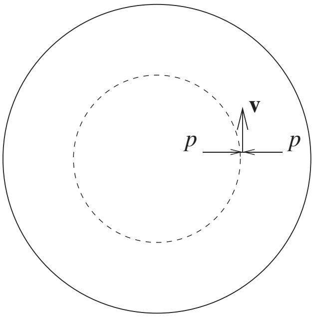



Figure 1.1 The radial pressure gradient is normal to circular surfaces and cannot change the moment of momentum of the fluid inside or outside the surface; it changes the direction of velocity **v** but not its modulus.

Any function $F(\mathbf{r}(t),t)$, like fluid temperature, varies for a moving particle in the same way, according to the chain rule of differentiation:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + (\mathbf{v}\cdot\nabla)F\ .$$

Writing now the second law of Newton for a unit mass of a fluid, we come to the equation derived by Euler (Berlin 1757; Petersburg 1759):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho}\ . \tag{1.2}$$

Before Euler, the acceleration of a fluid had been considered as due to the difference of the pressure exerted by the enclosing walls. Euler introduced the pressure field *inside* the fluid. For example, for the steadily rotating fluid shown in Figure 1.1, the acceleration vector $(\mathbf{v}\cdot\nabla)\mathbf{v}$ has a nonzero radial component $v^2/r$. The radial acceleration times the density gives the radial pressure gradient: $\mathrm{d}p/\mathrm{d}r = \rho v^2/r$.

We can also add an external body force per unit mass (for gravity $\mathbf{f} = \mathbf{g}$):

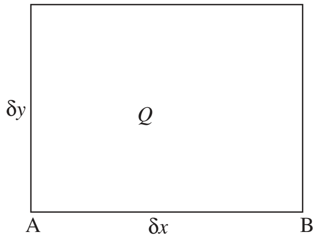
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{f}. \tag{1.3}$$

The term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ describes inertia and makes (1.3) nonlinear.

**Continuity equation.** This expresses conservation of mass. If $Q$ is the volume of a moving element then $\mathrm{d}\rho Q/\mathrm{d}t = 0$, that is

$$Q\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\mathrm{d}Q}{\mathrm{d}t} = 0. \tag{1.4}$$

The volume change can be expressed via $\mathbf{v}(\mathbf{r},t)$.




The horizontal velocity of the point B relative to the point A is $\delta x \partial v_x/\partial x$. After the time interval d$t$, the length of the edge AB is $\delta x(1+\mathrm{d}t\partial v_x/\partial x)$. Overall, after d$t$, one has the volume change

$$\mathrm{d}Q = \mathrm{d}t\,\delta x\delta y\delta z\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right) = \mathrm{d}t\,Q\,\mathrm{div}\,\mathbf{v} = \mathrm{d}t\,\frac{\mathrm{d}Q}{\mathrm{d}t}\ .$$

Substituting that into (1.4) and canceling (arbitrary) $Q$ we obtain the continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\,\mathrm{div}\,\mathbf{v} = \frac{\partial\rho}{\partial t}+(\mathbf{v}\cdot\nabla)\rho+\rho\,\mathrm{div}\,\mathbf{v} = \frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\mathbf{v}) = 0\ . \tag{1.5}$$

The last equation is almost obvious since for any *fixed volume of space* the decrease of the total mass inside, $-\int(\partial\rho/\partial t)\,\mathrm{d}V$, is equal to the flux $\oint\rho\mathbf{v}\cdot\mathrm{d}\mathbf{f} = \int\mathrm{div}(\rho\mathbf{v})\mathrm{d}V$.

**Entropy equation.** We now have four equations (1.3, 1.5) for five quantities $p,\rho,v_x,v_y,v_z$, so we need one extra equation. In deriving (1.3, 1.5) we have taken no account of energy dissipation, thus neglecting internal friction (viscosity) and heat exchange. A fluid without viscosity and thermal conductivity is called *ideal*. The motion of an ideal fluid is adiabatic, that is the entropy of any fluid particle remains constant: $\mathrm{d}s/\mathrm{d}t=0$, where $s$ is the entropy per unit mass. We can turn this equation into a continuity equation for the entropy density in space

$$\frac{\partial(\rho s)}{\partial t}+\mathrm{div}(\rho s\mathbf{v}) = 0\ . \tag{1.6}$$

Since entropy is a function of pressure and density then (1.6) is the needed extra relation between velocity, pressure and density. Different media differ by the form of the function $s(P,\rho)$.

**Boundary conditions.** At the boundaries of the fluid, the continuity equation (1.5) is replaced by the *boundary conditions*:

(1) On a fixed boundary, $v_n=0$;
(2) On a moving boundary between two immiscible fluids, $p_1=p_2$ and $v_{n1}=v_{n2}$.

These are particular cases of the general surface condition. Let $F(\mathbf{r},t)=0$ be the equation of the bounding surface. An absence of any fluid flow across the surface requires

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\frac{\partial F}{\partial t}+(\mathbf{v}\cdot\nabla)F=0\,,$$

which means, as we now know, the zero rate of $F$ variation for a fluid particle. For a stationary boundary, $\partial F/\partial t=0$ and $\mathbf{v}\perp\nabla F\Rightarrow v_n=0$.

### 1.1.3 Hydrostatics

A necessary and sufficient condition for fluid to be in a mechanical equilibrium follows from (1.3):

$$\nabla p=\rho\mathbf{f}\,. \tag{1.7}$$

Not every distribution of $\rho(\mathbf{r})$ could be in equilibrium since $\rho(\mathbf{r})\mathbf{f}(\mathbf{r})$ is not necessarily a gradient. If the force is potential, $\mathbf{f}=-\nabla\phi$, then taking the *curl* of (1.7) we get

$$\nabla\rho\times\nabla\phi=0.$$

This means that the gradients of $\rho$ and $\phi$ are parallel and their level surfaces coincide in equilibrium. The best-known example is gravity with $\phi=gz$ and $\partial p/\partial z=-\rho g$. For an incompressible fluid, it gives

$$p(z)=p(0)-\rho gz.$$

For an ideal gas under a homogeneous temperature, which has $p=\rho T/m$, one gets

$$\frac{\mathrm{d}p}{\mathrm{d}z}=-\frac{pgm}{T}\quad\Rightarrow\quad p(z)=p(0)\exp(-mgz/T).$$

For air at 0°C, $T/mg\simeq 8$ km. The Earth's atmosphere is described by neither a linear nor an exponential law because of an inhomogeneous temperature (Figure 1.2). Assuming a linear temperature decay, $T(z)=T_0-\alpha z$, one obtains a better approximation:

$$\frac{\mathrm{d}p}{\mathrm{d}z}=-\rho g=-\frac{pmg}{T_0-\alpha z}\,,$$
$$p(z)=p(0)(1-\alpha z/T_0)^{mg/\alpha},$$

which can be used not far from the surface with $\alpha\simeq 6.5°\mathrm{C\,km^{-1}}$.

Under gravity, density depends only on the distance from the Earth center (or locally on the vertical coordinate $z$) in a mechanical equilibrium. According

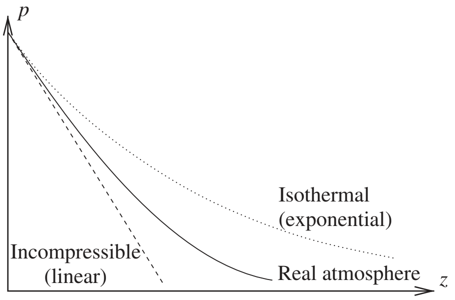



Figure 1.2 Pressure–height dependence for an incompressible fluid (broken line), isothermal gas (dotted line) and a real atmosphere (solid line).

to $\mathrm{d}p/\mathrm{d}z = -\rho g$, the pressure also depends only on $z$. Pressure and density determine temperature, which must then also be independent of the horizontal coordinates. Different temperatures at the same height, in particular nonuniform temperature of the Earth surface, necessarily produce fluid motion, which is why winds blow in the atmosphere and currents flow in the ocean. Another source of atmospheric flows is thermal convection due to a negative vertical temperature gradient. Let us derive the stability criterion for a fluid with a vertical profile $T(z)$. If a fluid element is shifted up adiabatically from $z$ by $\mathrm{d}z$, it keeps its entropy $s(z)$ but acquires the pressure $p' = p(z+\mathrm{d}z)$ so its new density is $\rho(s,p')$. For stability, this density must exceed the density of the displaced air at the height $z+\mathrm{d}z$, which has the same pressure but different entropy $s' = s(z+\mathrm{d}z)$. The condition for stability of the stratification is as follows:

$$\rho(p',s) > \rho(p',s') \Rightarrow \left(\frac{\partial\rho}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} < 0\,.$$

Entropy usually increases under expansion, $(\partial\rho/\partial s)_p < 0$, and for stability we must require $\mathrm{d}s/\mathrm{d}z > 0$. Entropy depends on $p, T$ which both decay with the height. Entropy decreases with cooling yet increases when $P$ decreases. To see which effect wins we compute:

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \left(\frac{\partial s}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial s}{\partial p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}z} = \frac{c_p}{T}\frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial V}{\partial T}\right)_p \frac{g}{V} > 0. \tag{1.8}$$

Here we used specific volume $V = 1/\rho$. For an ideal gas the coefficient of the thermal expansion gives $(\partial V/\partial T)_p = V/T$ and we end up with

$$\frac{g}{c_p} > -\frac{\mathrm{d}T}{\mathrm{d}z}. \tag{1.9}$$

Indeed, stability requires that the gain in potential energy $gdz$ must exceed the decrease in thermal energy $c_p dT$. For the Earth's atmosphere, $c_p \sim 10^3 \mathrm{J/kg^{-1}K^{-1}}$ and the convection threshold is $10^\circ\,\mathrm{C\,km^{-1}}$. The average gradient is $6.5^\circ\,\mathrm{C\,km^{-1}}$, that is, the entropy decreases with the height and the atmosphere is globally stable. However, local gradients vary very much depending on ground albedo, evaporation, etc., so that the atmosphere is often locally unstable with respect to thermal convection. The human body always excites convection in room-temperature air.[2]

Temperature decays with height only in the troposphere that is until about $-50^\circ$ C at 10–12 km, it is then constant up to about 35 km so that the pressure decays exponentially, eventually it grows in the stratosphere until about $0^\circ$ C at 50 km. Looking down from the plane flying above 10 km one often sees flat cloud top, particularly so-called anvil clouds, which is exactly where unstable air stratification below turns into stable above.

The convection stability argument applied to an incompressible fluid rotating with the angular velocity $\Omega(r)$ gives the Rayleigh's stability criterion, $\mathrm{d}(r^2\Omega)^2/\mathrm{d}r > 0$, which states that the angular momentum of the fluid $L = r^2|\Omega|$ must increase with the distance $r$ from the rotation axis.[3] Indeed, if a fluid element is shifted from $r$ to $r'$ it keeps its angular momentum $L(r)$, so that the local pressure gradient $\mathrm{d}p/\mathrm{d}r = \rho r' \Omega^2(r')$ must overcome the centrifugal force $\rho r'(L^2 r^4/r'^4)$.

### 1.1.4 Isentropic motion

The simplest motion corresponds to constant $s$ and allows for a substantial simplification of the Euler equation. Indeed, it would be convenient to represent $\nabla p/\rho$ as a gradient of some function. For this end, we need a function that depends on $p, s$, so that at $s = \text{const.}$ its differential is expressed solely via $\mathrm{d}p$. There exists the thermodynamic potential called *enthalpy*, defined as $W = E + pV$ per unit mass ($E$ is the internal energy of the fluid). For our purposes, it is enough to remember from thermodynamics the single relation $\mathrm{d}E = T\mathrm{d}s - p\mathrm{d}V$ so that $\mathrm{d}W = T\mathrm{d}s + V\mathrm{d}p$ (one can also show that $W = \partial(E\rho)/\partial\rho$). Since $s = \text{const.}$ for an isentropic motion and $V = \rho^{-1}$ for a unit mass, $\mathrm{d}W = \mathrm{d}p/\rho$ and, without body forces one has

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla W. \tag{1.10}$$

Such a gradient form will be used extensively for obtaining conservation laws, integral relations, etc. For example, we can use the vector identity $\mathbf{A}\times(\nabla\times\mathbf{B}) = \mathbf{A}\cdot(\nabla\mathbf{B}) - (\mathbf{A}\cdot\nabla)\mathbf{B}$) to represent

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=\nabla v^2/2-\mathbf{v}\times(\nabla\times\mathbf{v}),$$

and get

$$\frac{\partial\mathbf{v}}{\partial t}=\mathbf{v}\times(\nabla\times\mathbf{v})-\nabla(W+v^2/2). \tag{1.11}$$

The first term on the right-hand side is perpendicular to the velocity. To project (1.11) along the velocity and get rid of this term, we define a streamline as a line whose tangent is everywhere parallel to the instantaneous velocity. The streamlines are then determined by the relations

$$\frac{\mathrm{d}x}{v_x}=\frac{\mathrm{d}y}{v_y}=\frac{\mathrm{d}z}{v_z}\,.$$

Note that for time-dependent flows streamlines are different from particle trajectories: tangents to streamlines give velocities at a given time while tangents to trajectories give velocities at subsequent times. One records streamlines experimentally by seeding fluids with light-scattering particles; each particle produces a short trace on a short-exposure photograph, and the length and orientation of the trace indicates the magnitude and direction of the velocity. Streamlines can intersect only at a point of zero velocity called the stagnation point.

Let us now consider a steady flow, assuming $\partial\mathbf{v}/\partial t=0$, and take the component of (1.11) along the velocity at a point:

$$\frac{\partial}{\partial l}(W+v^2/2)=0\,. \tag{1.12}$$

We see that $W+v^2/2=E+p/\rho+v^2/2$ is constant along any given streamline, but may be different for different streamlines (Bernoulli 1738). Bernoulli theorem, of course, is a particular case of energy conservation. The change of the total energy density is not zero along the streamline but is equal to $P_2/\rho_2-P_1/\rho_1$ which is the work done. This is the reason $W$ rather $E$ enters the conservation law, as also discussed after (1.18). Alternatively, one may say that $W$ is a potential energy of a fluid particle, see (1.41) below. In a gravity field,

$$W+gz+v^2/2=\text{const.} \tag{1.13}$$

Without much exaggeration, one can say that most fluid-mechanics estimates use (1.12) or (1.13). Let us consider several applications of this useful relation.

Imagine that our spaceship suffered a meteorite attack that left holes in the walls of the cabin and the tank with liquid fuel. We need to estimate how fast we lose air from the cabin and fuel from the tank. Since there is vacuum outside, we can neglect thermal exchange and consider both flows isentropic. Liquid could be treated as incompressible, its internal energy $E$ is then constant without any external force. Bernoulli theorem then gives the limiting velocity with which such a liquid escapes from a large reservoir into vacuum:

$$v = \sqrt{2p_0/\rho} \, .$$

For water ($\rho = 10^3\,\mathrm{kg\,m^{-3}}$) at atmospheric pressure ($p_0 = 10^5\,\mathrm{N\,m^{-2}}$) one gets $v = \sqrt{200} \approx 14\,\mathrm{m\,s^{-1}}$.

For a gas, pressure drop must be accompanied by density change. The adiabatic law, $p/p_0 = (\rho/\rho_0)^\gamma$, gives the enthalpy as:

$$W = \int \frac{\mathrm{d}p}{\rho} = \frac{\gamma p}{(\gamma-1)\rho} \, .$$

The limiting velocity for the escape into vacuum can again be found from Bernoulli theorem:

$$\frac{\gamma p_0}{(\gamma-1)\rho} = \frac{v^2}{2} \quad \Rightarrow \quad v = \sqrt{\frac{2\gamma p_0}{(\gamma-1)\rho}},$$

The velocity is $\sqrt{\gamma/(\gamma-1)}$ times larger than for an incompressible fluid which corresponds to the limit $\gamma \gg 1$. The gas flows faster because the internal energy of the gas decreases as it flows, thus increasing the kinetic energy. We conclude that a meteorite-damaged spaceship loses the air from the cabin faster than the liquid fuel from the tank. We shall see later that $(\partial P/\partial \rho)_s = \gamma P/\rho$ is the sound velocity squared, $c^2$, so that $v = c\sqrt{2/(\gamma-1)}$. For an ideal gas with $n$ internal degrees of freedom, $W = E + p/\rho = nT/2m + T/m$ so that $\gamma = (2+n)/n$. For bi-atomic molecules $n = 5$ (3 translations and 2 rotations) at not very high temperature, when vibrations are not excited.

Another frequent occurrence is efflux from a small orifice under the action of gravity. Supposing the external pressure to be the same at the horizontal surface and at the orifice, we apply the Bernoulli relation to the streamline which originates at the upper surface with almost zero velocity and exits with velocity $v = \sqrt{2gh}$ (Torricelli 1643). The Torricelli formula is not of much use practically to calculate the rate of discharge, which in reality is not equal to the orifice area times $\sqrt{2gh}$, the fact known to wine merchants long before physicists. Indeed, streamlines converge from all sides toward the orifice so that the jet continues to converge for a while after coming out (Figure 1.3). Moreover, the converging motion makes the pressure in the interior of the jet somewhat greater than that at the surface (as is clear from the curvature of streamlines) so that the velocity in the interior is somewhat less than $\sqrt{2gh}$. The experiment shows that contraction ceases and the jet becomes cylindrical