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978-1-107-12922-1 — Nonlinear Mechanics of Shells and Plates in Composite, Soft and Biological Materials
Marco Amabili
Excerpt
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1 Classical Nonlinear Theories of Elasticity of Plates and Shells

1.1 Introduction

It is well known that certain elastic bodies may undergo large displacements while the strain at each point remains small. The classical theory of elasticity treats only problems in which displacements and their derivatives are small. Therefore, to treat such cases, it is necessary to introduce a theory of nonlinear elasticity with small strains. If the strains are small, the deformation in the neighbourhood of each point can be identified with a deformation to which the linear theory is applicable. This gives a rationale for adopting Hooke's stress–strain relations, and in the resulting nonlinear theory, large parts of the classical theory are preserved (Stoker 1968). However, the original and deformed configuration of a solid now cannot be assumed to be coincident, and the strains and stresses can be evaluated in the original undeformed configuration by using Lagrangian description, or in the deformed configuration by using Eulerian description (Fung 1965).

In this chapter, the classical geometrically nonlinear theories for rectangular plates, circular cylindrical shells, circular plates and spherical shells are derived, classical theories being those that neglect the shear deformation. Results are obtained in Lagrangian description, the effect of geometric imperfections is considered and the formulation of the elastic strain energy is also given. Classical theories for shells of any shape, as well as theories including shear deformation, are addressed in Chapters 2 and 3.

1.1.1 Literature Review

A short overview of some theories for geometrically nonlinear shells and plates will now be given. Some information is taken from the review by Amabili and Païdoussis (2003) and a more recent review by Alijani and Amabili (2014).

In the classical linear theory of plates, there are two fundamental methods for the solution of the problem. The first method was proposed by Cauchy (1828) and Poisson (1829), the second by Kirchhoff (1850). The method of Cauchy and Poisson is based on the expansion of displacements and stresses in the plate in power series of the distance z from the middle surface. Disputes concerning the convergence of these series and about the necessary boundary conditions made this method unpopular. Moreover, the method proposed by Kirchhoff has the advantage of introducing physical meaning into the theory of plates. Von Kármán (1910) extended this method to study finite deformation of plates, taking into account nonlinear terms. The nonlinear dynamic case was studied by Chu and Herrmann (1956), who were the pioneers in studying nonlinear vibrations of rectangular plates. In order to deal with thicker and laminated composite plates, the Reissner-Mindlin theory of plates (first-order shear deformation theory) was introduced to take into account transverse shear strains. Five variables are used in this theory to describe the deformation: three displacements of the middle surface and two rotations. The Reissner-Mindlin approach does not satisfy the transverse shear boundary conditions at the top and bottom surfaces of the plate because a constant shear angle through the thickness is assumed and plane sections remain plane after deformation. As a consequence of this approximation, the Reissner-Mindlin theory of plates requires shear correction factors for equilibrium considerations. For this reason, Reddy (1990) has developed a nonlinear plate theory that includes cubic terms (in the distance from the middle surface of the plate) in the in-plane displacement kinematics. This higher-order shear deformation theory satisfies zero transverse shear stresses at the top and bottom surfaces of the plate; up to cubic terms are retained in the expression of the shear, giving a parabolic shear strain distribution through the thickness, resembling with good approximation the results of three-dimensional elasticity. The same five variables of the Reissner-Mindlin theory are used to describe the kinematics in this higher-order shear deformation theory, but shear correction factors are not required.

Donnell (1934) established the nonlinear theory of circular cylindrical shells under the simplifying shallow-shell hypothesis. Because of its relative simplicity and practical accuracy, this theory has been widely used. The most frequently used form of Donnell's nonlinear shallow-shell theory (also referred to as Donnell-Mushtari-Vlasov theory) introduces a stress function in order to combine the three equations of equilibrium involving the shell displacements in the radial, circumferential and axial directions into two equations involving only the radial displacement w and the stress function F. This theory is accurate only for modes with: circumferential wavenumber n that are not small; specifically, $1/n^2 \ll 1$ must be satisfied, so that $n \geq 4$ or 5 is required in order to have fairly good accuracy. Donnell's nonlinear shallow-shell equations are obtained by neglecting the in-plane inertia, transverse shear deformation and rotary inertia, giving accurate results only for very thin shells. The predominant nonlinear terms are retained, but other secondary effects, such as the nonlinearities in curvature strains, are neglected; specifically, the curvature changes are expressed by linear functions of w only.

Von Kármán and Tsien (1941) performed a seminal study on the stability of axially loaded circular cylindrical shells, based on Donnell's nonlinear shallow-shell theory. In their book, Mushtari and Galimov (1957) presented nonlinear theories for moderate and large deformations of thin elastic shells. The nonlinear theory of shallow shells is also discussed in the book by Vorovich (1999), where the classical Russian studies, for example due to Mushtari and Vlasov, are presented.

Sanders (1963) developed a more refined nonlinear theory of shells, expressed in tensorial form. The same equations were obtained by Koiter (1966) around the same period, leading to the designation of these equations as the Sanders-Koiter equations. Later, this theory was reformulated in lines-of-curvature coordinates, that is, in a form that can be more suitable for applications; see, for example, Budiansky (1968), where only linear terms are given. According to the Sanders-Koiter theory, all three displacements are used in the equations of motion. Changes in curvature and torsion are linear according to both the Donnell and the Sanders-Koiter nonlinear theories (Yamaki 1984). The Sanders-Koiter theory gives accurate results for vibration amplitudes significantly larger than the shell thickness for thin shells (Amabili 2003).

Details on the aforementioned nonlinear shell theories may be found in Yamaki (1984) and Amabili (2003), with an introduction to another accurate theory called the modified Flügge nonlinear theory of shells, also referred to as the Flügge-Lur'e-Byrne nonlinear shell theory (Ginsberg 1973). The Flügge-Lur'e-Byrne theory is close to the general large deflection theory of thin shells developed by Novozhilov (1953) and differs only in terms for change in curvature and torsion.

Additional nonlinear shell theories were formulated by Naghdi and Nordgren (1963), using the Kirchhoff hypotheses, and by Libai and Simmonds (1988).

In order to treat moderately thick laminated shells, the nonlinear first-order shear deformation theory of shells was introduced by Reddy and Chandrashekhara (1985), which is based on the linear first-order shear deformation theory introduced by Reddy (1984). Five independent variables, three displacements and two rotations, are used to describe the shell deformation. This theory may be regarded as the thick-shell version of the Sanders theory for linear terms and of the Donnell nonlinear shell theory for nonlinear terms. A linear higher-order shear deformation theory of shells has been introduced by Reddy and Liu (1985); see also Reddy (2003). Dennis and Palazotto have extended this theory to nonlinear deformations (1990); see also Soldatos (1992). More refined geometrically nonlinear theories have been developed by Amabili and Reddy (2010) and Amabili (2015).

Shell theories taking into account thickness deformation are important for very large deformations of soft structures and allow to use three-dimensional constitutive equations. Some advanced shell theories that take shear and thickness deformation into account are those of Carrera et al. (2011), Alijani and Amabili (2014), Payette and Reddy (2014), Amabili (2015), and Gutiérrez Rivera et al. (2016).

The nonlinear mechanics of composite laminated shells has also been investigated by many authors. Librescu (1987) developed refined nonlinear theories for anisotropic laminated shells. Other theories applied to the dynamics of laminated shells have been developed, for example, by Tsai and Palazotto (1991), Pai and Nayfeh (1994), Kobayashi and Leissa (1995), Sansour et al. (1997), and Gummadi and Palazotto (1999). Nonlinear electromechanics of piezoelectric laminated shallow spherical shells was developed by Zhou and Tzou (2000).

1.2 Large Deflection of Rectangular Plates

1.2.1 Green's and Almansi Strain Tensors for Finite Deformation

It is assumed that a continuous body changes its configuration under physical actions and the change is continuous (no fractures are considered). A system of coordinates x_1, x_2, x_3 is chosen so that a point P of a body at a certain instant of time is described by the coordinates x_i ($i = 1,2,3$). At a later instant of time, the body has moved and deformed to a new configuration; the point P has moved to Q with coordinates a_i ($i = 1,2,3$) with

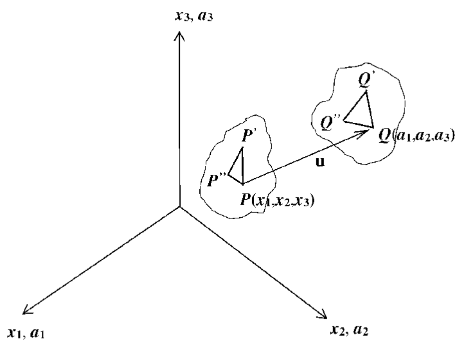

Figure 1.1 Body in the original configuration and after displacement **u**, which moves point P to Q.

respect to a new coordinate system a_1, a_2, a_3 (see Figure 1.1). Both coordinate systems are assumed to be the same rectangular Cartesian (rectilinear and orthogonal) coordinates for simplicity. The point transformation from P to Q is considered to be one-to-one, so that there is a unique inverse of the transformation. The functions x_i and a_i, describing the coordinates, are assumed to be continuous and differentiable, and the Jacobian determinant of the transformation is positive (i.e., a right-hand set of coordinates is transformed into another right-hand set) and does not vanish at any point. The displacement vector **u** is introduced having the following components:

$$u_i = a_i - x_i, \quad \text{for} \quad i = 1, \ldots, 3. \tag{1.1}$$

In the present book, the Lagrangian description of the continuous systems is used for convenience; therefore u_i are considered to be functions of x_i in order to evaluate the Lagrangian (or Green's) strain tensor – that is, the strains are referred to the original undeformed configuration.

If P, P' and P'' are three neighbouring points forming a triangle in the original configuration, and if they are transformed to points Q, Q' and Q'' in the deformed configuration, the change in the area and angles of the triangle are completely determined if the change in the length of the sides is known. However, the location of the triangle in space is not determined by the change of the sides. Similarly, if the change in length between any two arbitrary points of the body is known, the new configuration of the body is completely defined except for the location of the body in the space. Because interest here is on strains and these are related to stresses, attention is now focused on the change in distance between any two points of the body.

An infinitesimal line connecting point P (x_1, x_2, x_3) to a neighbourhood point P' (x_1+dx_1, x_2+dx_2, x_3+dx_3) is considered; the square of its length in the original configuration is given by

$$\overline{PP'}^2 = \mathrm{d}s_0^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2. \tag{1.2}$$

When, due to deformation, P and P' become Q (a_1, a_2, a_3) and Q' (a_1+da_1, a_2+da_2, a_3+da_3), respectively, the square of the distance is

$$\overline{QQ'}^2 = \mathrm{d}s^2 = \mathrm{d}a_1^2 + \mathrm{d}a_2^2 + \mathrm{d}a_3^2, \tag{1.3}$$

in the coordinate system a_i. The differentials $\mathrm{d}a_i$ can be transformed in the original coordinate system x_i:

$$\mathrm{d}a_i = \frac{\partial a_i}{\partial x_1}\mathrm{d}x_1 + \frac{\partial a_i}{\partial x_2}\mathrm{d}x_2 + \frac{\partial a_i}{\partial x_3}\mathrm{d}x_3. \tag{1.4}$$

Therefore, by using equation (1.4), equation (1.3) is transformed into

$$\mathrm{d}s^2 = \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial a_k}{\partial x_i}\frac{\partial a_k}{\partial x_j}\mathrm{d}x_i\mathrm{d}x_j. \tag{1.5}$$

The difference between the squares of the length of the elements may be written in the following form by using equations (1.2) and (1.5):

$$\mathrm{d}s^2 - \mathrm{d}s_0^2 = \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial a_k}{\partial x_i}\frac{\partial a_k}{\partial x_j} - \delta_{ij}\right)\mathrm{d}x_i\mathrm{d}x_j, \tag{1.6}$$

where δ_{ij} is the Kronecker delta, equal to 1 if $i = j$ and otherwise equal to zero. By definition, Green's strain tensor ε_{ij} is obtained as

$$\mathrm{d}s^2 - \mathrm{d}s_0^2 = 2\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{ij}\mathrm{d}x_i\mathrm{d}x_j, \tag{1.7}$$

and therefore is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\sum_{k=1}^{3}\frac{\partial a_k}{\partial x_i}\frac{\partial a_k}{\partial x_j} - \delta_{ij}\right). \tag{1.8}$$

By using equation (1.1), the following expression is obtained:

$$\frac{\partial a_k}{\partial x_i} = \frac{\partial u_k}{\partial x_i} + \delta_{ki}. \tag{1.9}$$

Finally, by substituting equation (1.9) into equation (1.8), Green's strain tensor is expressed as

$$\varepsilon_{ij} = \frac{1}{2}\left[\left(\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_i} + \delta_{ki}\right)\left(\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_j} + \delta_{kj}\right) - \delta_{ij}\right] = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right). \tag{1.10}$$

In unabridged notation (x, y, z for x_1, x_2, x_3), the typical formulations are obtained:

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_3}{\partial x}\right)^2\right], \tag{1.11a}$$

$$\gamma_{xy} = \frac{1}{2}\left[\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \left(\frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y}\right)\right], \tag{1.11b}$$

where the usual symbol γ_{xy} has been used instead of ε_{xy} in equation (1.11b). Equation (1.10) shows that Green's strain tensor is symmetric. To visualize the physical significance of the Green's strain, it is considered a test specimen that does not change direction after deformation. Being that ds is the length of the specimen after elongation, and ds_0 its initial length, the Green's strain in the direction of the elongation x (same direction before and after deformation) is given by

$$\varepsilon_{xx} = \frac{\mathrm{d}\,s^2 - \mathrm{d}\,s_0^2}{2\mathrm{d}\,s^2}.$$

If the Eulerian description of the continuous systems is used, u_i are considered functions of a_i in order to evaluate the Eulerian (usually referred to as the Almansi) strain tensor (i.e., the strains are referred to the deformed configuration). With analogous mathematical development, the Almansi strain tensor is given by

$$\varepsilon_{ij}^{(E)} = \frac{1}{2}\left[\delta_{ij} - \left(-\sum_{k=1}^{3}\frac{\partial u_k}{\partial a_i} + \delta_{ki}\right)\left(-\sum_{k=1}^{3}\frac{\partial u_k}{\partial a_j} + \delta_{kj}\right)\right] = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} - \sum_{k=1}^{3}\frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right), \tag{1.12}$$

which is also symmetric; the superscript denotes Eulerian. Considering the same test specimen, the Eulerian strain in the direction of the elongation x is given by

$$\varepsilon_{xx}^{(E)} = \frac{\mathrm{d}\,s^2 - \mathrm{d}\,s_0^2}{2\mathrm{d}\,s_0^2}.$$

Therefore, the only difference with respect to the Green's strain is that the deformation is now divided by twice the square of the deformed length instead of the original length.

1.2.2 Strains for Finite Deflection of Rectangular Plates: Von Kármán Theory

The displacements of a generic point of the middle surface of the plate are indicated by u, v and w in x, y and z direction, respectively; the corresponding displacements of a generic point of the plate at distance z from the middle surface are denoted by u_1, u_2 and u_3, as shown in Figure 1.2(a).

When the plate deflection w is of the same order of magnitude as the plate thickness h, results obtained by using linear theories become quite inaccurate. Here a theory for large deflections (large refers to the fact that w is not small compared to h, so that the original and deformed configurations are different) of plates is developed in rectangular coordinates (suitable for rectangular plates of in-plane dimensions a and b, see Figure 1.2(b)). In the theory, the following hypotheses are made (Fung 1965):

(H1) The plate is thin: $h \ll a$, $h \ll b$.
(H2) The magnitude of deflection w is of the same order as the thickness h of the plate and, therefore, small compared to the plate dimensions a and b for (H1): $|w| = O(h)$.
(H3) The slope is small at each point: $|\partial w/\partial x| \ll 1$, $|\partial w/\partial y| \ll 1$.
(H4) All strain components are small so that linear elasticity can be applied.
(H5) Kirchhoff's hypotheses hold; that is, stresses in the direction normal to the plate middle surface are negligible, and strains vary linearly within the plate thickness.

These hypotheses are a good approximation for thin plates. However, in the presence of external loads orthogonal to the plate surface, stresses in the normal direction arise, even if they are generally of some order of magnitude smaller than other stresses.

(H6) For von Kármán's hypothesis, the in-plane displacements u and v are infinitesimal, and in the strain–displacement relations, only those nonlinear terms that depend on w need to be retained. All other nonlinear terms may be neglected.

Hypothesis (H6) can be removed in order to get more accurate nonlinear plate theories.

Figure 1.3 shows that the deformed configuration of the plate differs significantly from the original one. The Lagrangian description of the plate is used so that the plate surfaces are always designated as $z = \pm h/2$; a right-handed rectangular Cartesian reference system (O; x, y, z) is used, with the x, y plane coinciding with the middle surface of the plate in its initial, undeformed configuration and the z axis normal to it. In the Lagrangian description, Green's strain tensor, referred to in the initial configuration, is used; it is given by equations (1.10) and (1.11). By using hypothesis (H5) we have

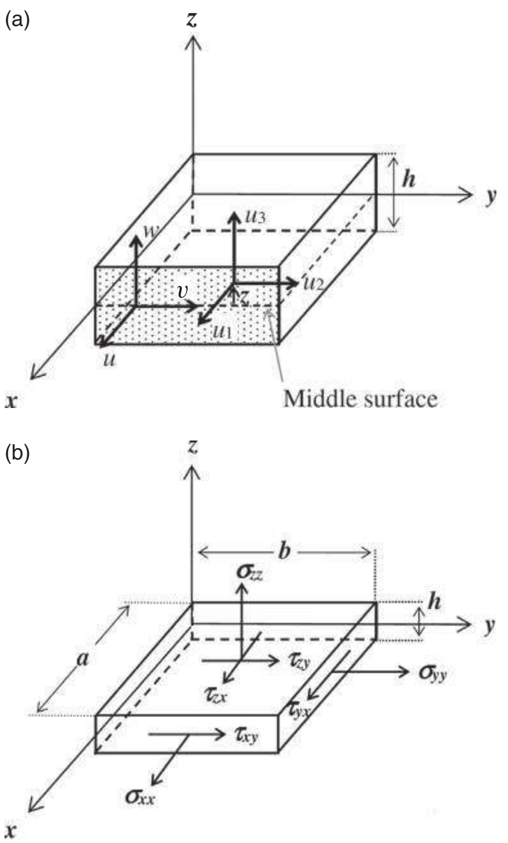

Figure 1.2 Rectangular plate. (a) Symbols used for displacements of middle surface and generic points; (b) symbols used for dimensions and second Piola-Kirchhoff stresses.

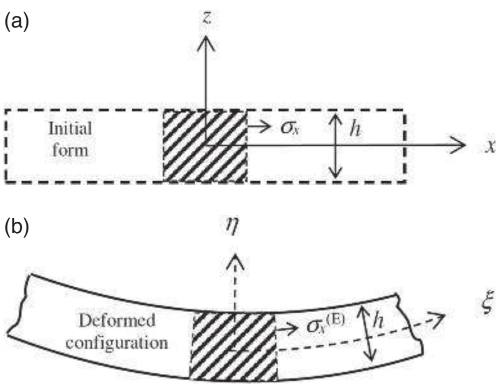

Figure 1.3 Cross-section of the rectangular plate. (a) Initial configuration; σ_x is the normal second Piola-Kirchhoff stress; (b) deformed configuration; $\sigma_x^{(E)}$ is the normal Eulerian stress.

$$u_1 = u(x, y) - z\frac{\partial w}{\partial x}, \tag{1.13}$$

$$u_2 = v(x, y) - z\frac{\partial w}{\partial y}, \tag{1.14}$$

$$u_3 = w(x, y). \tag{1.15}$$

Equations (1.13–1.15) are linear expressions. In particular, these equations are obtained as follows: hypothesis (H5) requires that strains vary linearly within the plate thickness and second Piola-Kirchhoff stresses in the direction normal to the plate middle surface are negligible (i.e., $\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$, where σ_{ij} is the normal stress acting on the surface normal to i in direction j, and τ_{ij} is the tangential stress as shown in Figure 1.2(b)); therefore, by using linear elasticity,

$$\sigma_{xx} = a_1(x, y) + b_1(x, y)z, \tag{1.16}$$

$$\sigma_{yy} = a_2(x, y) + b_2(x, y)z, \tag{1.17}$$

$$\varepsilon_{zz} \simeq \frac{\partial u_3}{\partial z} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right), \tag{1.18}$$

where E is Young's modulus and ν is Poisson's ratio. In the last equation, the linearized expression of ε_{zz} is used (this is the reason for using $\simeq$ instead of =) as a consequence of equations (1.13–1.15) being linear. Integration of equation (1.18) gives

$$u_3 = w(x, y) - \frac{\nu}{E}[a_1(x, y) + a_2(x, y)]z - \frac{\nu}{E}[b_1(x, y) + b_2(x, y)]\frac{z^2}{2}, \tag{1.19}$$

where the term w is the integration constant. The last two terms on the right-hand side are in general very small for thin plates, as a consequence of ν/E being a very small number, and they can be neglected; therefore, equation (1.15) is verified, and $\varepsilon_{zz} \simeq 0$. Also, for hypothesis (H5), the following expressions are obtained:

$$\gamma_{zx} \simeq \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) = \frac{\tau_{zx}}{E} - \frac{\nu}{E}\left(\tau_{zy} + \sigma_{zz}\right) = 0, \tag{1.20}$$

$$\gamma_{zy} \simeq \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right) = 0. \tag{1.21}$$

Inserting $u_3 = w(x, y)$ from equation (1.15) into equations (1.20) and (1.21) and integrating, equations (1.13) and (1.14) are obtained.

Equations (1.13–1.15) are inserted into Green's strain tensor, equations (1.10) and (1.11), in order to obtain the strain–displacement relations for a plate in rectangular coordinates. The strain components ε_{xx}, ε_{yy} and γ_{xy} at an arbitrary point of the plate are related to the middle surface strains $\varepsilon_{x,0}$, $\varepsilon_{y,0}$ and $\gamma_{xy,0}$ and to the changes in the curvature and torsion of the middle surface k_x, k_y and k_{xy} by the following three relations:

$$\varepsilon_{xx} = \varepsilon_{x,0} + z k_x, \tag{1.22}$$

$$\varepsilon_{yy} = \varepsilon_{y,0} + z k_y, \tag{1.23}$$

$$\gamma_{xy} = \gamma_{xy,0} + z k_{xy}, \tag{1.24}$$

where z is, as usual, the distance of the arbitrary point of the plate from the middle surface. If the von Kármán hypothesis (H6) is used, the following expressions for the middle surface strains and the changes in the curvature and torsion of the middle surface are obtained, namely

$$\varepsilon_{x,0} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{1.25}$$

$$\varepsilon_{y,0} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \tag{1.26}$$

$$\gamma_{xy,0} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \tag{1.27}$$

$$k_x = -\frac{\partial^2 w}{\partial x^2}, \tag{1.28}$$

$$k_y = -\frac{\partial^2 w}{\partial y^2}, \tag{1.29}$$

$$k_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}. \tag{1.30}$$

These expressions are in general accurate enough for moderately large vibrations of plates. If more accurate expressions are needed, hypothesis (H6) can be removed and equations (1.25–1.30) can be computed, retaining all the nonlinear terms.

1.2.3 Geometric Imperfections

Initial geometric imperfections of the rectangular plate associated with zero initial stress are denoted by normal displacement w_0; in-plane initial imperfections are neglected. Therefore, equation (1.15) is replaced by

$$u_3 = w(x,y) + w_0(x,y). \tag{1.31}$$

Substituting equation (1.31) into Green's strain tensor and neglecting all terms that depend on w_0 only (as they are associated with initial deformation and therefore with zero stress), the following expressions are obtained, replacing equations (1.25–1.27):

$$\varepsilon_{x,0} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x}, \tag{1.32}$$

$$\varepsilon_{y,0} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial w_0}{\partial y}, \tag{1.33}$$

$$\gamma_{xy,0} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w}{\partial y}. \tag{1.34}$$

Equations (1.28–1.30) are unchanged by the presence of the imperfection w_0. It can be observed that, in the presence of geometric imperfections, the plate is no longer perfectly flat.

1.2.4 Eulerian, Lagrangian and Second Piola-Kirchhoff Stress Tensors

In the deformed configuration of a body, the equations of equilibrium of an infinitesimal parallelepiped with surfaces parallel to the coordinate planes (see Figure 1.4) are given by

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ji}^{(E)}}{\partial a_j} + X_i = 0, \qquad \text{for } i = 1, \ldots, 3, \tag{1.35}$$

where $\sigma_{ji}^{(E)}$ is the Eulerian stress tensor (for simplicity, the symbol σ is used here also for the tangential stresses instead of τ). That is, the stresses referred to the deformed configuration, $a_{j,}$ are the coordinates describing the deformed configuration, and X_i are the body forces, including inertia, per unit volume. The first subscript of $\sigma_{ji}^{(E)}$ indicates the normal to the plane on which the stress is acting, while the second indicates the stress component, as shown in Figure 1.4. It must be observed that in literature, you can find a reverse notation, which gives the transpose of the stress tensor defined here.

The Eulerian stress tensor is also referred to as the Cauchy stress tensor or the true stress tensor. The Eulerian stresses are symmetric; that is, $\sigma_{ji}^{(E)} = \sigma_{ij}^{(E)}$. Equation (1.35) shows that, in the Eulerian description, the dynamic equations are simple, but the kinematics is complex; for this reason, it is convenient to find the relationships to transform the Eulerian stresses in the original undeformed configuration in order to use