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Introduction and Constitutive Equations For Linearly Elastic Materials

Plates are structural elements given by a flat surface with a given thickness h. The flat surface is the middle surface of the plate; the upper and lower surfaces delimiting the plate are at distance h/2 from the middle surface. The thickness is small compared with the in-plane dimensions and can be either constant or variable. Thin plates are very stiff for in-plane loads, but they are quite flexible in bending. Many applications of plates, made of extremely different materials, can be found in engineering. For example, very thin circular plates are used in computer hard disk drives; rectangular and trapezoidal plates can be found in the wing skin, horizontal tail surfaces, flaps and vertical fins of aircraft; cantilever rectangular plates are used as nano-resonators for drug detection; and clamped circular thin nano-plates in graphene are tested to be used as nano-devices for pressure measurement.

On the other hand, shells present a curved middle surface. These structures are abundantly present in nature. In fact, because of the curvature of the middle surface, shells are very stiff for both in-plane and bending loads; therefore, they can span over large areas by using a minimum amount of material. In the human bodies, arteries and the tympanic membrane are shells.

Shells are largely used in engineering; some shell structures are impressive and beautiful. Shell structural elements are largely present in spacecraft, aeronautics and sport cars, where the use of composite materials is becoming very significant (see Figure 1). Functionally graded materials also have potential for applications in space and nuclear engineering.

One of the main targets in the design of shell structural elements is to make the thickness as small as possible to spare material and to make the structure light. The analysis of shells has difficulty related to the curvature, which is also the reason for the carrying load capacity of these structures. In fact, a change of the curvature can give a totally different strength. Moreover, because of the optimal distribution of material, shells collapse for buckling much before the failure strength of the material is reached. For their thin nature, they can present large displacements, with respect to the shell thickness, associated to small strains before collapse. This is the rationale for using a nonlinear shell and plate theory for studying shell stability.

Shells are often subjected to dynamic loads that cause vibrations; vibration amplitudes of the order of the shell thickness can be easily reached in many applications. Therefore, a nonlinear shell theory should be applied. However, the most challenging shells and plates to study are those made of soft biological materials. In fact, due to the very large deformations, the presence of reinforcing fibers specifically oriented, and the 2

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Introduction



Figure 1 Boeing 787, the first commercial airplane made mostly of composite materials (image from www.boeing.com – courtesy of Boeing).



Figure 2 Layers of an artery (image from www.full-health.com).

physically nonlinear characteristics of their material, their study is complex (see Figure 2). Biological materials can be considered hyperelastic for static loads and viscoelastic in dynamics.

The book is organized in 14 chapters. Chapter 1 discusses classical nonlinear theories for rectangular and circular plates, circular cylindrical and spherical shells. Classical shell theories for doubly curved shallow shells and for shells of arbitrary shape are discussed in Chapter 2 together with an exact description of the pressure load. Composite and functionally graded materials are introduced in Chapter 3 with advanced nonlinear shell theories that include shear deformation, rotary inertia and thermal loads. Chapter 4 deals with an advanced shell theory that takes into account the thickness deformation. This is significant in case of large deformations, as those observed for soft materials. The first four chapters are self-contained with the full development of the theories under clear hypotheses and limitations. They present material that is usually spread in several articles and books with different approaches and symbols. The shell theories are expressed in lines-of-curvature coordinates, which is the form suitable for applications and computer implementation. In some cases, improved formulations, suitable for thick shells and large rotations, have been developed.

Hyperelastic materials, which are absolutely necessary to model soft biological tissues as the human skin, ligaments or arteries, are treated in Chapter 5. The nonlinear

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I.1 Equations for linearly elastic materials

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dynamics, stability, bifurcation analysis and modern computational tools are introduced in Chapter 6. The Galerkin method and the energy approach that leads to the Lagrange equations of motion are introduced here. Chapter 7 addresses damping, linear and nonlinear viscoelasticity. Advanced nonlinear damping models are introduced based on the theory of viscoelasticity.

Chapter 7 closes the most general part of the book, with the following seven chapters addressing specific problems and applications.

Linear and nonlinear vibrations of rectangular plates with different boundary conditions are studied in Chapter 8. Both isotropic and laminated composite materials are treated. Plates in functionally graded materials are instead investigated in Chapter 9. Chapter 10 addresses rectangular plates in rubber and biological materials. Linear and nonlinear vibrations of simply supported circular cylindrical shells in isotropic and laminated composite materials are studied in Chapter 11. Nonlinear vibrations of circular cylindrical shells with different boundary conditions are addressed in Chapter 12. The specific problem of the response of the human aorta subjected to static and dynamic blood pressure is the subject of Chapter 13, where the effect of residual stresses is taken into account. Nonlinear vibrations of doubly curved shells with rectangular base, including spherical and hyperbolic paraboloidal shells, are investigated in Chapter 14. Both classical and first-order shear deformation theories are used to study nonlinear vibrations of laminated composite shells. Static buckling, including the effect of geometric imperfections is also addressed. The static buckling and nonlinear vibrations of circular cylindrical shells under axial loads are investigated in Chapter 15, with particular attention to geometric imperfections and period-doubling dynamic bifurcations.

I.1 Constitutive Equations for Linearly Elastic Materials

The constitutive equations characterize the individual material and its reaction to applied loads. In this section, the constitutive equations for linearly elastic material are introduced, where linear means that the material undergoes small deformations. In a subsequent chapter (Chapter 5), more complicated constitutive equations for nonlinear materials are introduced, and they are particularly suitable for large strains, which characterize soft materials.

A homogeneous solid has material properties that are the same throughout the body. A heterogeneous body has properties that are a function of the position. An anisotropic body has material properties that are different in different directions at the same point. An isotropic solid has the same properties in any direction at any point. An anisotropic or isotropic material can be homogeneous or heterogeneous.

A solid is said to be perfectly elastic when it returns to its original shape after removing the loads that caused the deformation, and there is a one-to-one correspondence between the state of stress and the state of strain. This excludes creep - i.e., strain at constant stress and stress relaxation at constant strain. In this chapter, it is assumed that the material is perfectly elastic.

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The constitutive equations for linear elasticity in three dimensions are

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \varepsilon_{kl}, \qquad (I.1)$$

where σ_{ij} and ε_{kl} are the components of the second-order stress and strain tensors, respectively. Here the first subscript indicates the plane orthogonal to that axis on which the stress or strain acts, and the second subscript indicates the direction. Instead, C_{ijkl} are the 81 components of the fourth-order elasticity modulus tensor. Due to the symmetry, only 36 constants are independent.

In case of isotropic linearly elastic materials, it is obtained that

$$\sigma_{ij} = \lambda \,\delta_{ij} \sum_{k=1}^{3} \varepsilon_{kk} + 2\mu \,\varepsilon_{ij},\tag{I.2}$$

in which λ and μ are the two Lamé elastic constants and δ_{ij} is the Kronecker delta. The relationship with the shear modulus *G*, Young's modulus *E*, Poisson's ratio *v* and bulk modulus κ are

$$G = \mu, \qquad E = \frac{G(3\lambda + 2G)}{\lambda + G}, \qquad \nu = \frac{\lambda}{2(\lambda + G)}, \qquad \kappa = \lambda + \frac{2}{3}\mu$$
 (I.3a-d)

I.1.1 3-D Constitutive Equations for a Layer within a Laminated Shell

The generalized Hooke's law relates the six components of stress to the six components of strain (see Figure 3) as Reddy (2007) provides

$$\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{cases}^{(k)} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix}^{(k)} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{cases},$$
(I.4)

where σ_{xx} , σ_{yy} and σ_{zz} are the orthogonal normal stresses in *x*, *y* and *z* direction, respectively, τ_{yz} , τ_{xz} and τ_{xy} are the shear stresses acting on the planes orthogonal to *y*, *x*, and *x*, respectively, and in direction *z*, *z* and *y*, respectively, and the superscript (*k*)



Figure 3 Stress components.

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refers to the *k*th layer within a laminate. This generic material needs 36 coefficients c_{ij} to be described.

Under the assumption that exists a potential function $W = W(\varepsilon_{ij})$, named the strain energy density function, whose derivative with respect to a strain component gives the corresponding stress component – i.e., $\sigma_{ij} = \partial W/\partial \varepsilon_{ij}$ – the 6 × 6 matrix **C** on the lefthand side of equation (I.4) becomes symmetric. If the strain energy density function exists, then the material is termed hyperelastic and the number of independent coefficients in the matrix **C** is reduced to 21.

In case of orthotropic material –i.e., when the material presents three mutually orthogonal symmetry planes – the three-dimensional stresses and strains in the material principal coordinates (x, y, z) in the *k*th layer of a laminated shell are linked by the relationship

$$\begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{cases} = \begin{bmatrix} 1/E_x & -v_{yx}/E_y & -v_{zx}/E_z & 0 & 0 & 0 \\ -v_{xy}/E_x & 1/E_y & -v_{zy}/E_z & 0 & 0 & 0 \\ -v_{xz}/E_x & -v_{yz}/E_y & 1/E_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{xz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{xy} \end{bmatrix}^{(k)} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}^{(k)},$$
(I.5)

where E_x , E_y and E_z are the Young's moduli in *x*, *y* and *z* direction, respectively, and G_{xy} , G_{xz} and G_{yz} are the shear moduli in *x*–*y*, *x*–*z* and *y*–*z* directions, respectively; v_{ij} are the Poisson's ratios.

The transverse isotropy assumption with respect to planes orthogonal to the *x* axis – i.e., assuming fibers in the direction parallel to axis *x* – is then introduced so that $E_y = E_z$, $G_{xy} = G_{xz}$, $v_{xy} = v_{xz}$, $v_{yx} = v_{zx}$ and $v_{yz} = v_{zy}$. Then only 9 independent coefficients are used in the matrix **C**. It can be observed that in the case of Poisson's ratios all equal to 0.5 (incompressible material), the matrix on the right-hand side of equation (I.5) becomes singular and cannot be inverted. In fact, for incompressible material, the strains ε_{xx} , ε_{yy} and ε_{zz} are not independent but linked by a relationship. In other cases, inverting equation (I.5) and keeping into account the assumption of transverse isotropy, the stress–strain relations for the *k*th orthotropic lamina of the shell in the material principal coordinates (*x*, *y*, *z*) are obtained (Reddy 2007)

$$\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{cases}^{(k)} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{xz} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{xy} \end{bmatrix}^{(k)} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{cases},$$
(I.6)

and the coefficients c_{ij} are given by

$$c_{11} = \frac{E_x (1 - v_{yz})}{(1 - v_{yz} - 2v_{xy}v_{yx})}, \quad c_{12} = c_{21} = c_{13} = c_{31} = \frac{v_{xy}E_y}{(1 - v_{yz} - 2v_{xy}v_{yx})} = \frac{v_{yx}E_x}{(1 - v_{yz} - 2v_{xy}v_{yx})},$$
(I.7a.b)

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$$c_{22} = c_{33} = \frac{(1 - v_{xy}v_{yx})E_y}{(1 + v_{yz})(1 - v_{yz} - 2v_{xy}v_{yx})}, \quad c_{23} = c_{32} = \frac{(v_{yz} + v_{xy}v_{yx})E_y}{(1 + v_{yz})(1 - v_{yz} - 2v_{xy}v_{yx})},$$
(I.7c,d)
$$v_{ij}E_j = v_{ji}E_i.$$
(I.7e)

In particular, equation (I.6) gives

$$\sigma_{zz} = \frac{v_{xy}E_y}{(1 - v_{yz} - 2v_{xy}v_{yx})}\varepsilon_{xx} + \frac{(v_{yz} + v_{xy}v_{yx})E_y}{(1 + v_{yz})(1 - v_{yz} - 2v_{xy}v_{yx})}\varepsilon_{yy} + \frac{(1 - v_{xy}v_{yx})E_y}{(1 + v_{yz})(1 - v_{yz} - 2v_{xy}v_{yx})}\varepsilon_{zz}.$$
(I.8)

I.1.2 Constitutive Equations in Case of Negligible Transverse Normal Stress

It is assumed that the transverse normal stress $\sigma_{zz} = 0$; i.e., it is negligible. In general, it is verified that σ_{zz} is small compared to the transverse shear stresses τ_{xz} and τ_{yz} , except near the shell edges, so that the hypothesis is a good approximation of the actual behaviour of moderately thick shells and plates. The stresses and strains in this case, in the material principal coordinates, in the *k*th layer of a laminated shell are linked by the relationship (Reddy 2007)

$$\begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{cases} = \begin{bmatrix} 1/E_x & -v_{21}/E_y & 0 & 0 & 0 \\ -v_{12}/E_x & 1/E_y & 0 & 0 & 0 \\ 0 & 0 & 1/G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 1/G_{xz} & 0 \\ 0 & 0 & 0 & 0 & 1/G_{xy} \end{bmatrix}^{(k)} \begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{cases}, \quad (I.9)$$

which can be inverted to give the stress-strain relations

$$\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{cases}^{(k)} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & G_{yz} & 0 & 0 \\ 0 & 0 & 0 & G_{xz} & 0 \\ 0 & 0 & 0 & 0 & G_{xy} \end{bmatrix}^{(k)} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{cases},$$
(I.10)

*(*1)

 $\langle 1 \rangle$

and the coefficients c_{ij} are given for a lamina by

$$c_{11} = \frac{E_x}{1 - v_{xy}v_{yx}}, \qquad c_{12} = c_{21} = \frac{E_y v_{xy}}{1 - v_{xy}v_{yx}}, \qquad c_{22} = \frac{E_y}{1 - v_{xy}v_{yx}}, \qquad v_{ij}E_j = v_{ji}E_i.$$
(I.11a-d)

Equations (I.11a–d) are obtained (1) under the transverse isotropy assumption with respect to planes orthogonal to the *x* axis – i.e., assuming fibers in the direction parallel to axis *x*, so that $E_y = E_z$, $G_{xy} = G_{xz}$ and $v_{xy} = v_{xz}$ – and (2) solving the constitutive equations for ε_{zz} as function of ε_{xx} and ε_{yy} and then eliminating it.

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I.1.3 Constitutive Equations in Case of Plane Stress (Classical Theories of Plates and Shells)

In the case of plain stress, $\sigma_3 = \tau_{13} = \tau_{23} = 0$, and equation (I.10) can be simplified into

$$\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{cases}^{(k)} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}^{(k)} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{cases},$$
(I.12)

and the coefficients c_{ij} are given for a lamina by

$$c_{11} = \frac{E_x}{1 - v_{xy}v_{yx}}, \qquad c_{12} = c_{21} = \frac{E_y v_{xy}}{1 - v_{xy}v_{yx}}, \qquad c_{22} = \frac{E_y}{1 - v_{xy}v_{yx}}, \qquad v_{xy}E_y = v_{yx}E_x.$$
(I.13a-d)

Equations (I.13a–d) are obtained under the transverse isotropy assumption with respect to planes orthogonal to the x axis. This formulation of the constitutive equations is used in conjunction with the classical theories of plates and shells.

In the case of homogeneous and isotropic material, which is characterized by only two coefficients, the Young modulus E and the Poisson's ratio v, the constitutive equations can be simplified into

$$\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{cases} = \begin{bmatrix} \frac{E}{1 - v^2} & \frac{vE}{1 - v^2} & 0 \\ \frac{vE}{1 - v^2} & \frac{E}{1 - v^2} & 0 \\ 0 & 0 & \frac{E}{2(1 + v)} \end{bmatrix} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{cases},$$
(I.14)

which gives

$$\sigma_{xx} = \frac{E}{1 - v^2} \left(\varepsilon_{xx} + v \varepsilon_{yy} \right), \qquad \sigma_{yy} = \frac{E}{1 - v^2} \left(\varepsilon_{yy} + v \varepsilon_{xx} \right), \qquad \tau_{xy} = \frac{E}{2(1 + v)} \gamma_{xy}.$$
(L15)

In equation (I.14), the following condition for the shear modulus, valid for homogeneous and isotropic materials, has been used:

$$G = \frac{E}{2(1+\nu)}.$$
 (I.16)

References

Y. C. FUNG 1965 Foundations of Solid Mechanics. Prentice-Hall, Englewood Cliffs, NJ, USA.

J. N. REDDY 2007 *Theory and Analysis of Elastic Plates and Shells*, 2nd edition, CRC Press, Boca Raton, FL, USA.

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1 Classical Nonlinear Theories of Elasticity of Plates and Shells

1.1 Introduction

It is well known that certain elastic bodies may undergo large displacements while the strain at each point remains small. The classical theory of elasticity treats only problems in which displacements and their derivatives are small. Therefore, to treat such cases, it is necessary to introduce a theory of nonlinear elasticity with small strains. If the strains are small, the deformation in the neighbourhood of each point can be identified with a deformation to which the linear theory is applicable. This gives a rationale for adopting Hooke's stress–strain relations, and in the resulting nonlinear theory, large parts of the classical theory are preserved (Stoker 1968). However, the original and deformed configuration of a solid now cannot be assumed to be coincident, and the strains and stresses can be evaluated in the original undeformed configuration by using Lagrangian description, or in the deformed configuration by using Eulerian description (Fung 1965).

In this chapter, the classical geometrically nonlinear theories for rectangular plates, circular cylindrical shells, circular plates and spherical shells are derived, classical theories being those that neglect the shear deformation. Results are obtained in Lagrangian description, the effect of geometric imperfections is considered and the formulation of the elastic strain energy is also given. Classical theories for shells of any shape, as well as theories including shear deformation, are addressed in Chapters 2 and 3.

1.1.1 Literature Review

A short overview of some theories for geometrically nonlinear shells and plates will now be given. Some information is taken from the review by Amabili and Païdoussis (2003) and a more recent review by Alijani and Amabili (2014).

In the classical linear theory of plates, there are two fundamental methods for the solution of the problem. The first method was proposed by Cauchy (1828) and Poisson (1829), the second by Kirchhoff (1850). The method of Cauchy and Poisson is based on the expansion of displacements and stresses in the plate in power series of the distance z from the middle surface. Disputes concerning the convergence of these series and about the necessary boundary conditions made this method unpopular. Moreover, the method proposed by Kirchhoff has the advantage of introducing physical meaning into the theory of plates. Von Kármán (1910) extended this method to study finite deformation of plates, taking into account nonlinear terms. The nonlinear dynamic case was studied

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by Chu and Herrmann (1956), who were the pioneers in studying nonlinear vibrations of rectangular plates. In order to deal with thicker and laminated composite plates, the Reissner-Mindlin theory of plates (first-order shear deformation theory) was introduced to take into account transverse shear strains. Five variables are used in this theory to describe the deformation: three displacements of the middle surface and two rotations. The Reissner-Mindlin approach does not satisfy the transverse shear boundary conditions at the top and bottom surfaces of the plate because a constant shear angle through the thickness is assumed and plane sections remain plane after deformation. As a consequence of this approximation, the Reissner-Mindlin theory of plates requires shear correction factors for equilibrium considerations. For this reason, Reddy (1990) has developed a nonlinear plate theory that includes cubic terms (in the distance from the middle surface of the plate) in the in-plane displacement kinematics. This higherorder shear deformation theory satisfies zero transverse shear stresses at the top and bottom surfaces of the plate; up to cubic terms are retained in the expression of the shear, giving a parabolic shear strain distribution through the thickness, resembling with good approximation the results of three-dimensional elasticity. The same five variables of the Reissner-Mindlin theory are used to describe the kinematics in this higher-order shear deformation theory, but shear correction factors are not required.

Donnell (1934) established the nonlinear theory of circular cylindrical shells under the simplifying shallow-shell hypothesis. Because of its relative simplicity and practical accuracy, this theory has been widely used. The most frequently used form of Donnell's nonlinear shallow-shell theory (also referred to as Donnell-Mushtari-Vlasov theory) introduces a stress function in order to combine the three equations of equilibrium involving the shell displacements in the radial, circumferential and axial directions into two equations involving only the radial displacement w and the stress function F. This theory is accurate only for modes with: circumferential wavenumber n that are not small; specifically, $1/n^2 \ll 1$ must be satisfied, so that $n \ge 4$ or 5 is required in order to have fairly good accuracy. Donnell's nonlinear shallow-shell equations are obtained by neglecting the in-plane inertia, transverse shear deformation and rotary inertia, giving accurate results only for very thin shells. The predominant nonlinear terms are retained, but other secondary effects, such as the nonlinearities in curvature strains, are neglected; specifically, the curvature changes are expressed by linear functions of w only.

Von Kármán and Tsien (1941) performed a seminal study on the stability of axially loaded circular cylindrical shells, based on Donnell's nonlinear shallow-shell theory. In their book, Mushtari and Galimov (1957) presented nonlinear theories for moderate and large deformations of thin elastic shells. The nonlinear theory of shallow shells is also discussed in the book by Vorovich (1999), where the classical Russian studies, for example due to Mushtari and Vlasov, are presented.

Sanders (1963) developed a more refined nonlinear theory of shells, expressed in tensorial form. The same equations were obtained by Koiter (1966) around the same period, leading to the designation of these equations as the Sanders-Koiter equations. Later, this theory was reformulated in lines-of-curvature coordinates, that is, in a form that can be more suitable for applications; see, for example, Budiansky (1968), where only linear terms are given. According to the Sanders-Koiter theory, all three displacements are used in the equations of motion. Changes in curvature and torsion are linear

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according to both the Donnell and the Sanders-Koiter nonlinear theories (Yamaki 1984). The Sanders-Koiter theory gives accurate results for vibration amplitudes significantly larger than the shell thickness for thin shells (Amabili 2003).

Details on the aforementioned nonlinear shell theories may be found in Yamaki (1984) and Amabili (2003), with an introduction to another accurate theory called the modified Flügge nonlinear theory of shells, also referred to as the Flügge-Lur'e-Byrne nonlinear shell theory (Ginsberg 1973). The Flügge-Lur'e-Byrne theory is close to the general large deflection theory of thin shells developed by Novozhilov (1953) and differs only in terms for change in curvature and torsion.

Additional nonlinear shell theories were formulated by Naghdi and Nordgren (1963), using the Kirchhoff hypotheses, and by Libai and Simmonds (1988).

In order to treat moderately thick laminated shells, the nonlinear first-order shear deformation theory of shells was introduced by Reddy and Chandrashekhara (1985), which is based on the linear first-order shear deformation theory introduced by Reddy (1984). Five independent variables, three displacements and two rotations, are used to describe the shell deformation. This theory may be regarded as the thick-shell version of the Sanders theory for linear terms and of the Donnell nonlinear shell theory for nonlinear terms. A linear higher-order shear deformation theory of shells has been introduced by Reddy and Liu (1985); see also Reddy (2003). Dennis and Palazotto have extended this theory to nonlinear theories have been developed by Amabili and Reddy (2010) and Amabili (2015).

Shell theories taking into account thickness deformation are important for very large deformations of soft structures and allow to use three-dimensional constitutive equations. Some advanced shell theories that take shear and thickness deformation into account are those of Carrera et al. (2011), Alijani and Amabili (2014), Payette and Reddy (2014), Amabili (2015), and Gutiérrez Rivera et al. (2016).

The nonlinear mechanics of composite laminated shells has also been investigated by many authors. Librescu (1987) developed refined nonlinear theories for anisotropic laminated shells. Other theories applied to the dynamics of laminated shells have been developed, for example, by Tsai and Palazotto (1991), Pai and Nayfeh (1994), Kobayashi and Leissa (1995), Sansour et al. (1997), and Gummadi and Palazotto (1999). Nonlinear electromechanics of piezoelectric laminated shallow spherical shells was developed by Zhou and Tzou (2000).

1.2 Large Deflection of Rectangular Plates

1.2.1 Green's and Almansi Strain Tensors for Finite Deformation

It is assumed that a continuous body changes its configuration under physical actions and the change is continuous (no fractures are considered). A system of coordinates x_1 , x_2 , x_3 is chosen so that a point *P* of a body at a certain instant of time is described by the coordinates x_i (*i* = 1,2,3). At a later instant of time, the body has moved and deformed to a new configuration; the point *P* has moved to *Q* with coordinates a_i (*i* = 1,2,3) with