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Uwe Franz and Nicolas Privault
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1

Boson Fock space

You don't know who he was? Half the particles in the universe obey him!

(Reply by a physics professor when a student asked who Bose was.)

We start by introducing the elementary boson Fock space together with its canonically associated creation and annihilation operators on a space of square-summable sequences, and in the more general setting of Hilbert spaces. The boson Fock space is a simple and fundamental quantum model which will be used in preliminary calculations of Gaussian moments on the boson Fock space, based on the commutation and duality relations satisfied by the creation and annihilation operators. Those calculations will also serve as a motivation for the general framework of the subsequent chapters.

1.1 Annihilation and creation operators

Consider the space of square-summable sequences

$$\ell^2 := \Gamma(\mathbb{C}) = \left\{ f : \mathbb{N} \to \mathbb{C} : \sum_{k=0}^{\infty} |f(k)|^2 < \infty \right\}$$

with the inner product

$$\langle f, g \rangle_{\ell^2} := \sum_{k=0}^{\infty} \overline{f(k)} g(k), \qquad f, g \in \ell^2,$$

and orthonormal basis $(e_n)_{n\in\mathbb{N}}$ given by the Kronecker symbols

$$e_n(k) := \delta_{k,n} = \begin{cases} 1 & k = n, \\ 0 & k \neq n, \end{cases}$$

$k, n \in \mathbb{N}$.

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Definition 1.1.1 *Let $\sigma > 0$. The annihilation and creation operators are the linear operators a^- and a^+ implemented on ℓ^2 by letting*

$$a^+ e_n := \sigma\sqrt{n+1}\, e_{n+1}, \qquad a^- e_n := \sigma\sqrt{n}\, e_{n-1}, \qquad n \in \mathbb{N}.$$

Note that the above definition means that $a^- e_0 = 0$.

The sequence space ℓ^2 endowed with the annihilation and creation operators a^- and a^+ is called the *boson (or bosonic) Fock space*. In the physical interpretation of the boson Fock space, the vector e_n represents a physical n-particle state. The term "boson" refers to the Bose–Einstein statistics and in particular to the possibility for n particles to share the same state e_n, and Fock spaces are generally used to model the quantum states of identical particles in variable number.

As a consequence of Definition 1.1.1 the *number operator* a° defined as $a^\circ := a^+ a^-$ has eigenvalues given by

$$a^\circ e_n = a^+ a^- e_n = \sigma^2 \sqrt{n} a^+ e_{n-1} = n\sigma^2 e_n, \qquad n \in \mathbb{N}. \tag{1.1}$$

Noting the relation

$$a^- a^+ e_n = \sigma\sqrt{n+1} a^- e_{n+1} = \sigma^2 (n+1) e_n,$$

in addition to (1.1), we deduce the next proposition.

Proposition 1.1.2 *We have the commutation relation*

$$[a^+, a^-] e_n = \sigma^2 e_n, \qquad n \in \mathbb{N}.$$

Quantum physics provides a natural framework for the use of the non-commutative operators a^- and a^+, by connecting them with the statistical intuition of probability. Indeed, the notion of physical measurement is noncommutative in nature; think, *e.g.*, of measuring the depth of a pool *vs.* measuring water temperature: each measurement will perturb the next one in a certain way, thus naturally inducing noncommutativity. In addition, noncommutativity gives rise to the impossibility of making measurements with infinite precision, and the physical interpretation of quantum mechanics is essentially probabilistic as a given particle only has a *probability density* of being in a given state/location. In the sequel we take $\sigma = 1$.

Given $f = (f(n))_{n\in\mathbb{N}}$ and $g = (g(n))_{n\in\mathbb{N}}$ written as

$$f = \sum_{n=0}^{\infty} f(n) e_n \quad \text{and} \quad g = \sum_{n=0}^{\infty} g(n) e_n,$$

we have

$$a^+ f = \sum_{n=0}^{\infty} f(n) a^+ e_n = \sum_{n=0}^{\infty} f(n)\sqrt{n+1}\, e_{n+1} = \sum_{n=1}^{\infty} f(n-1)\sqrt{n}\, e_n$$

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and

$$a^- f = \sum_{n=0}^\infty f(n) a^- e_n = \sum_{n=1}^\infty f(n)\sqrt{n}\, e_{n-1} = \sum_{n=0}^\infty f(n+1)\sqrt{n+1}\, e_n,$$

hence we have

$$(a^+ f)(n) = \sqrt{n} f(n-1), \quad \text{and} \quad (a^- f)(n) = \sqrt{n+1} f(n+1). \qquad (1.2)$$

This shows the following duality relation between a^- and a^+.

Proposition 1.1.3 *For all $f, g \in \ell^2$ with finite support in $\mathbb{N}$ we have*

$$\langle a^- f, g\rangle_{\ell^2} = \langle f, a^+ g\rangle_{\ell^2}.$$

Proof: By (1.2) we have

$$\begin{aligned}\langle a^- f, g\rangle_{\ell^2} &= \sum_{n=0}^\infty \overline{(a^- f)(n)} g(n) \\ &= \sum_{n=0}^\infty \sqrt{n+1}\, \overline{f(n+1)} g(n) \\ &= \sum_{n=1}^\infty \sqrt{n}\, \overline{f(n)} g(n-1) \\ &= \sum_{n=1}^\infty \overline{f(n)} (a^+ g)(n) \\ &= \langle f, a^+ g\rangle_{\ell^2}.\end{aligned}$$

□

We also define the *position* and *momentum* operators

$$Q := a^- + a^+ \qquad \text{and} \qquad P := i(a^+ - a^-),$$

which satisfy the commutation relation

$$[P, Q] = PQ - QP = -2I_d.$$

To summarise the results of this section, the Hilbert space $H = \ell^2$ with inner product $\langle\cdot,\cdot\rangle_{\ell^2}$ has been equipped with two operators a^- and a^+, called *annihilation and creation operators* and acting on the elements of H such that

a) a^- and a^+ are dual of each other in the sense that

$$\langle a^- u, v\rangle_{\ell^2} = \langle u, a^+ v\rangle_{\ell^2},$$

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and this relation will also be written as $(a^+)^* = a^-$, with respect to the inner product $\langle \cdot, \cdot \rangle_{\ell^2}$.

b) the operators a^- and a^+ satisfy the commutation relation

$$[a^+, a^-] = a^+a^- - a^-a^+ = \sigma^2 I_d,$$

where Id is the identity operator.

1.2 Lie algebras on the boson Fock space

In this section we characterise the Lie algebras made of linear mappings

$$Y: \ell^2 \longmapsto \ell^2,$$

written on the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of the boson Fock space ℓ^2 as

$$Ye_n = \gamma_n e_{n+1} + \epsilon_n e_n + \eta_n e_{n-1}, \qquad n \in \mathbb{N}, \tag{1.3}$$

where $\gamma_n, \epsilon_n, \eta_n \in \mathbb{C}$, with $\eta_0 = 0$ and $\gamma_n \neq 0$, $n \in \mathbb{N}$. We assume that Y is Hermitian, *i.e.*, $Y^* = Y$, or equivalently

$$\bar{\gamma}_n = \eta_{n+1} \quad \text{and} \quad \epsilon_n \in \mathbb{R}, \qquad n \in \mathbb{N}.$$

For example, the *position* and *moment* operators

$$Q := a^- + a^+ \qquad \text{and} \qquad P := i(a^+ - a^-)$$

can be written as

$$Qe_n = a^-e_n + a^+e_n = \sqrt{n}e_{n-1} + \sqrt{n+1}e_{n+1},$$

i.e., $\gamma_n = \sqrt{n+1}$, $\epsilon_n = 0$, and $\eta_n = \sqrt{n}$, while

$$Pe_n = i(a^+e_n - a^-e_n) = i\sqrt{n+1}e_{n+1} - i\sqrt{n}e_{n-1},$$

i.e., $\gamma_n = i\sqrt{n+1}$, $\epsilon_n = 0$, and $\eta_n = -i\sqrt{n}$.

In the sequel we consider the sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials given by

$$P_n(Y) := \sum_{k=0}^{n} \alpha_{k,n} Y^k, \qquad n \in \mathbb{N}.$$

Proposition 1.2.1 *The condition*

$$e_n = P_n(Y)e_0, \qquad n \in \mathbb{N}, \tag{1.4}$$

defines a unique sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials that satisfy the three-term recurrence relation

$$xP_n(x) = \gamma_n P_{n+1}(x) + \epsilon_n P_n(x) + \eta_n P_{n-1}(x), \qquad n \in \mathbb{N}, \tag{1.5}$$

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from which the sequence $(P_n)_{n\in\mathbb{N}}$ *can be uniquely determined based on the initial condition* $P_{-1} = 0$, $P_1 = 1$.

Proof: The relation (1.3) and the condition (1.4) show-that

$$\begin{aligned} YP_n(Y)e_0 &= \gamma_n P_{n+1}(Y)e_0 + \epsilon_n P_n(Y)e_0 + \eta_n P_{n-1}(Y)e_0 \\ &= \gamma_n e_{n+1} + \epsilon_n e_n + \eta_n e_{n-1}, \end{aligned}$$

which implies the recurrence relation (1.5). □

For example, the monomial Y^n satisfies

$$\langle e_n, Y^n e_0\rangle_{\ell^2} = \gamma_0 \cdots \gamma_{n-1}, \qquad n \in \mathbb{N},$$

hence since $\gamma_n \neq 0$, $n \in \mathbb{N}$, we have in particular

$$\begin{aligned} 1 &= \langle e_n, e_n\rangle_{\ell^2} \\ &= \langle e_n, P_n(Y)e_0\rangle_{\ell^2} \\ &= \sum_{k=0}^{n} \alpha_{k,n} \langle e_n, Y^k e_0\rangle_{\ell^2} \\ &= \alpha_{n,n}\langle e_n, Y^n e_0\rangle_{\ell^2} \\ &= \alpha_{n,n}\gamma_1 \cdots \gamma_n, \qquad n \in \mathbb{N}. \end{aligned}$$

In the case where $Y = Q$ is the position operator, imposing the relation

$$e_n = P_n(Q)e_0, \qquad n \in \mathbb{N},$$

i.e., (1.4), shows that

$$QP_n(Q)e_0 = \sqrt{n+1}P_{n+1}(Q)e_0 + \sqrt{n}P_{n-1}(Q)e_0,$$

hence the three-term recurrence relation (1.5) reads

$$xP_n(x) = \sqrt{n+1}P_{n+1}(x) + \sqrt{n}P_{n-1}(x),$$

for $n \in \mathbb{N}$, with initial condition $P_{-1} = 0$, $P_1 = 1$, hence $(P_n)_{n\in\mathbb{N}}$ is the family of normalised Hermite polynomials, cf. Section 12.1.

Definition 1.2.2 *By a probability law of Y in the* fundamental state e_0 *we will mean a probability measure* μ *on* $\mathbb{R}$ *such that*

$$\int_{\mathbb{R}} x^n \mu(dx) = \langle e_0, Y^n e_0\rangle_{\ell^2}, \qquad n \in \mathbb{N},$$

which is also called the spectral measure *of Y evaluated in the state* $Y \mapsto \langle e_0, Y e_0\rangle_{\ell^2}$.

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In this setting the *moment generating function* defined as

$$t \longmapsto \langle e_0, e^{tY} e_0 \rangle_{\ell^2}$$

will be used to determine the probability law μ of Y in the state e_0.

We note that in this case the polynomials $P_n(x)$ are orthogonal with respect to $\mu(dx)$, since

$$\begin{aligned}\int_{-\infty}^{\infty} P_n(x)P_m(x)\mu(dx) &= \langle e_0, P_n(Y)P_m(Y)e_0\rangle_{\ell^2} \\ &= \langle P_n(Y)e_0, P_m(Y)e_0\rangle_{\ell^2} \\ &= \langle e_n, e_m\rangle_{\ell^2} \\ &= \delta_{n,m}, \qquad n, m \in \mathbb{N}.\end{aligned}$$

1.3 Fock space over a Hilbert space

More generally, the boson Fock space also admits a construction upon any real separable Hilbert space $\mathfrak{h}$ with complexification $\mathfrak{h}_{\mathbb{C}}$, and in this more general framework it will simply be called the Fock space.

The basic structure and operators of the Fock space over $\mathfrak{h}$ are similar to those of the simple boson Fock space, however it allows for more degrees of freedom. The boson Fock space ℓ^2 defined earlier corresponds to the symmetric Fock space over the one-dimensional real Hilbert space $\mathfrak{h} = \mathbb{R}$. We will use the conjugation operator

$$\bar{\ }: \mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}$$

on the complexification

$$\mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \oplus i\mathfrak{h} = \{h_1 + ih_2 : h_1,\ h_2 \in \mathfrak{h}\},$$

of $\mathfrak{h}$, defined by letting

$$\overline{h_1 + ih_2} := h_1 - ih_2, \qquad h_1, h_2 \in \mathfrak{h}.$$

This conjugate operation satisfies

$$\langle \bar{h}, \bar{k}\rangle_{\mathfrak{h}_{\mathbb{C}}} = \overline{\langle h, k\rangle}_{\mathfrak{h}_{\mathbb{C}}} = \langle k, h\rangle_{\mathfrak{h}_{\mathbb{C}}}, \qquad h, k \in \mathfrak{h}_{\mathbb{C}}.$$

The elements of $\mathfrak{h}$ are characterised by the property $\bar{h} = h$, and we will call them real. The next definition uses the notion of the symmetric tensor product "$\circ$" in Hilbert spaces.

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Definition 1.3.1 *The* symmetric Fock space *over* $\mathfrak{h}_{\mathbb{C}}$ *is defined by the direct sum*

$$\Gamma_s(\mathfrak{h}) = \bigoplus_{n\in\mathbb{N}} \mathfrak{h}_{\mathbb{C}}^{\circ n}.$$

We denote by $\mathbf{\Omega} := \mathbf{1} + 0 + \cdots$ the vacuum vector in $\Gamma_s(\mathfrak{h})$. The symmetric Fock space is isomorphic to the complexification of the Wiener space $L^2(\Omega)$ associated to $\mathfrak{h}$ in Section 9.2.

The exponential vectors

$$\mathcal{E}(f) := \sum_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}}, \qquad f \in \mathfrak{h}_{\mathbb{C}},$$

are total in $\Gamma_s(\mathfrak{h})$, and their scalar product in $\Gamma_s(\mathfrak{h})$ is given by

$$\langle \mathcal{E}(k_1), \mathcal{E}(k_2)\rangle_{\mathfrak{h}_{\mathbb{C}}} = e^{\langle k_1,k_2\rangle_{\mathfrak{h}_{\mathbb{C}}}}.$$

1.3.1 Creation and annihilation operators on $\Gamma_s(\mathfrak{h})$

The annihilation, creation, position, and momentum operators $a^-(h)$, $a^+(h)$, $Q(h)$, $P(h)$, $h \in \mathfrak{h}$, can be defined as unbounded and closed operators on the Fock space over $\mathfrak{h}$, see, *e.g.*, [17, 79, 87]. The creation and annihilation operators $a^+(h)$ and $a^-(h)$ are mutually adjoint, and the position and momentum operators

$$Q(h) = a^-(\overline{h}) + a^+(h) \quad \text{and} \quad P(h) = i\big(a^-(\overline{h}) - a^+(h)\big)$$

are self-adjoint if $h \in \mathfrak{h}$ is real. The commutation relations of creation, annihilation, position, and momentum are

$$\begin{cases} [a(h), a^+(k)] = \langle h, k\rangle_{\mathfrak{h}_{\mathbb{C}}}, \\ [a(h), a(k)] = [a^+(h), a^+(k)] = 0, \\ [Q(h), Q(k)] = [P(h), P(k)] = 0, \\ [P(h), Q(k)] = 2i\langle \overline{h}, k\rangle_{\mathfrak{h}_{\mathbb{C}}}. \end{cases}$$

The operators $a^-(h), a^+(h), Q(h), P(h)$ are unbounded, but their domains contain the exponential vectors $\mathcal{E}(f), f \in \mathfrak{h}_{\mathbb{C}}$. We will need to compose them with bounded operators on $\Gamma_s(\mathfrak{h})$, and in order to do so we will adopt the

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following convention. Let

$$\begin{aligned}&\mathcal{L}\big(\mathcal{E}(\mathfrak{h}_{\mathbb{C}}),\Gamma_s(\mathfrak{h})\big)\\&=\Big\{B\in\mathrm{Lin}\,\big(\mathrm{span}(\mathcal{E}(\mathfrak{h}_{\mathbb{C}})),\Gamma_s(\mathfrak{h})\big):\exists B^*\in\mathrm{Lin}\,\big(\mathrm{span}(\mathcal{E}(\mathfrak{h}_{\mathbb{C}})),\Gamma_s(\mathfrak{h})\big)\\&\qquad\text{such that }\big\langle\mathcal{E}(f),B\mathcal{E}(g)\big\rangle_{\mathfrak{h}_{\mathbb{C}}}=\big\langle B^*\mathcal{E}(f),\mathcal{E}(g)\big\rangle_{\mathfrak{h}_{\mathbb{C}}}\text{ for all }f,g\in\mathfrak{h}_{\mathbb{C}}\Big\},\end{aligned}$$

denote the space of linear operators that are defined on the exponential vectors and that have an "adjoint" that is also defined on the exponential vectors. Obviously the operators $a^-(h), a^+(h), Q(h), P(h), U(h_1, h_2)$ belong to $\mathcal{L}\big(\mathcal{E}(\mathfrak{h}_{\mathbb{C}}),\Gamma_s(\mathfrak{h})\big)$. We will say that an expression of the form

$$\sum_{j=1}^n X_jB_jY_j,$$

with $X_1,\ldots,X_n,Y_1,\ldots,Y_n\in\mathcal{L}\big(\mathcal{E}(\mathfrak{h}_{\mathbb{C}}),\Gamma_s(\mathfrak{h})\big)$ and $B_1,\ldots,B_n\in\mathcal{B}\big(\Gamma_s(\mathfrak{h})\big)$ defines a bounded operator on $\Gamma_s(\mathfrak{h})$, if there exists a bounded operator $M\in\mathcal{B}\big(\Gamma_s(\mathfrak{h})\big)$ such that

$$\big\langle\mathcal{E}(f),M\mathcal{E}(g)\big\rangle_{\mathfrak{h}_{\mathbb{C}}}=\sum_{j=1}^n\big\langle X_j^*\mathcal{E}(f),B_jY_j\mathcal{E}(g)\big\rangle_{\mathfrak{h}_{\mathbb{C}}}$$

holds for all $f,g\in\mathfrak{h}_{\mathbb{C}}$. If it exists, this operator is unique because the exponential vectors are total in $\Gamma_s(\mathfrak{h})$, and we will then write

$$M=\sum_{j=1}^n X_jB_jY_j.$$

1.3.2 Weyl operators

The Weyl operators $U(h_1,h_2)$ are defined by

$$U(h_1,h_2)=\exp\big(iP(h_1)+iQ(h_2)\big)=\exp\Big(i\big(a^-(\overline{h_2}-i\overline{h_1})+a^+(h_2-ih_1)\big)\Big),$$

and they satisfy

$$U(h_1,h_2)U(k_1,k_2)=\exp i\big(\langle\overline{h}_2,k_1\rangle_{\mathfrak{h}_{\mathbb{C}}}-\langle\overline{h}_1,k_2\rangle_{\mathfrak{h}_{\mathbb{C}}}\big)U(h_1+h_2,k_1+k_2).$$

Furthermore, we have $U(h_1,h_2)^*=U(-\overline{h}_1,-\overline{h}_2)$ and $U(h_1,h_2)^{-1}=U(-h_1,-h_2)$. We see that $U(h_1,h_2)$ is unitary, if h_1 and h_2 are real. These operators act on the vacuum state $\mathbf{\Omega}=\mathcal{E}(0)$ as

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$$U(h_1,h_2)\mathbf{\Omega} = \exp\left(-\frac{\langle \overline{h}_1, h_1\rangle_{\mathfrak{h}_\mathbb{C}} + \langle \overline{h}_2, h_2\rangle_{\mathfrak{h}_\mathbb{C}}}{2}\right)\mathcal{E}(h_1+ih_2)$$

and on the exponential vectors $\mathcal{E}(f)$ as

$$\begin{aligned}U(h_1,h_2)&\mathcal{E}(f)\\ &= \exp\left(-\langle \overline{f}, h_1+ih_2\rangle_{\mathfrak{h}_\mathbb{C}} - \frac{\langle \overline{h}_1, h_1\rangle_{\mathfrak{h}_\mathbb{C}} + \langle \overline{h}_2, h_2\rangle_{\mathfrak{h}_\mathbb{C}}}{2}\right)\mathcal{E}(f+h_1+ih_2).\end{aligned}$$

Exercises

Exercise 1.1 *Moments of the normal distribution.*

In this exercise we consider an example in which the noncommutativity property of a^- and a^+ naturally gives rise to a fundamental example of probability distribution, *i.e.*, the normal distribution.

In addition to that we will assume the existence of a unit vector $\mathbf{1} \in \mathfrak{h}$ (fundamental or empty state) such that $a^-\mathbf{1} = 0$ and $\langle \mathbf{1}, \mathbf{1}\rangle_\mathfrak{h} = 1$. In particular, this yields the rule

$$\langle a^+u, \mathbf{1}\rangle_\mathfrak{h} = \langle u, a^-\mathbf{1}\rangle_\mathfrak{h} = 0.$$

Based on this rule, check by an elementary computation that the first four moments of the centered $\mathcal{N}(0,\sigma^2)$ can be recovered from $\langle Q^n\mathbf{1}, \mathbf{1}\rangle_\mathfrak{h}$ with $n = 1,2,3,4$.

In the following chapters this problem will be addressed in a systematic way by considering other algebras and probability distributions as well as the problem of *joint distributions* such as the distribution of the couple (P,Q).

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Real Lie algebras

> Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine".
>
> *(M. Atiyah,* Collected works.*)*

In this chapter we collect the definition and properties of the real Lie algebras that will be needed in the sequel. We consider in particular the Heisenberg–Weyl Lie algebra $\mathfrak{hw}$, the oscillator Lie algebra $\mathfrak{osc}$, and the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$, $so(2)$, and $so(3)$ as particular cases. Those examples and their relationships with classical probability distributions will be revisited in more details in the subsequent chapters.

2.1 Real Lie algebras

Definition 2.1.1 *A* Lie algebra $\mathfrak{g}$ *over a field* $\mathbb{K}$ *is a* $\mathbb{K}$*-vector space with a linear map* $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$ *called* Lie bracket *that satisfies the following two properties.*

1. *Anti-symmetry: for all* $X, Y \in \mathfrak{g}$*, we have*

$$[X,Y] = -[Y,X].$$

2. *Jacobi identity: for all* $X, Y, Z \in \mathfrak{g}$*, we have*

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0.$$

For $\mathbb{K}=\mathbb{R}$, we call $\mathfrak{g}$ a *real Lie algebra*, for $\mathbb{K}=\mathbb{C}$ a *complex Lie algebra*.

Definition 2.1.2 *Let* $\mathfrak{g}$ *be a complex Lie algebra. An* involution *on* $\mathfrak{g}$ *is a conjugate linear map* $*\colon \mathfrak{g}\longrightarrow\mathfrak{g}$ *such that*

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