1

The central force problem

Moment. Das Moment des Eindrucks, den ein Mann auf das gemeine Volk macht, ist ein Produkt aus dem Wert des Rocks in den Titel.

Georg Christoph Lichtenberg

We start by dealing with an idealised problem, where a body of mass m – more precisely, a dimensionless particle – is moving in euclidean 3-space $\mathbb{R}^3$ subject to an attractive force directed towards a fixed centre, which for convenience we place at the origin $\mathbf{0} \in \mathbb{R}^3$. To begin with, we allow more general forces than the one described by Newton's law of gravitation; the force may even be repelling in some regions of space.

The central force problem Find solutions of the differential equation[1]

$$m\ddot{\mathbf{r}} = -mf(\mathbf{r}) \cdot \frac{\mathbf{r}}{r}, \tag{CFP}$$

where $f\colon \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}$ is a given continuous function.

By a solution of (CFP) we mean a C^2-map $\mathbf{r}\colon I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$, defined on some interval $I \subset \mathbb{R}$, that satisfies the differential equation (CFP).

The equation (CFP) expresses the requirement that the force act along the line joining the body and the centre $\mathbf{0}$, and that it be proportional to the mass m, so that in fact m is irrelevant to the solution of (CFP). The sign in (CFP) has been chosen in such a way that the force is attracting at points $\mathbf{r} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ where $f(\mathbf{r}) > 0$, which is the case we usually consider, and repelling where $f(\mathbf{r}) < 0$.

As is standard in the theory of differential equations, the second-order differential equation (CFP) can be rewritten as a system of first-order equations

[1] All equations with individual labels are listed in the Index under 'equations'.

by introducing the velocity $\mathbf{v} := \dot{\mathbf{r}}$ as an additional variable:

$$\left.\begin{array}{rcl}\dot{\mathbf{r}} & = & \mathbf{v}\\ \dot{\mathbf{v}} & = & -f(\mathbf{r})\cdot\dfrac{\mathbf{r}}{r}.\end{array}\right\}\qquad \text{(CFP}'\text{)}$$

Any C^2-solution $t \mapsto \mathbf{r}(t)$ of (CFP) gives rise to a C^1-solution $t \mapsto (\mathbf{r}(t), \dot{\mathbf{r}}(t))$ of (CFP′). Conversely, the first component of a C^1-solution $t \mapsto (\mathbf{r}(t), \mathbf{v}(t))$ of (CFP′) is a solution of (CFP); notice that the equations (CFP′) imply that this first component is actually of class C^2.

The advantage of the formulation (CFP′) is that we can think of solutions as integral curves of the vector field $\mathbf{X}$ on $(\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times \mathbb{R}^3$ defined by

$$\mathbf{X}(\mathbf{r}, \mathbf{v}) = (\mathbf{v}, -f(\mathbf{r}) \cdot \mathbf{r}/r).$$

This point of view will become relevant in Sections 3.1 and 6.2, for instance.

A first step towards understanding a system of differential equations is to ask whether there are any preserved quantities. In the physical context considered here, these will be referred to as **constants of motion**. By this we mean functions of $\mathbf{r}$ and $\mathbf{v}$ that are constant along integral curves of (CFP′). In this chapter we shall meet two such constants of motion: the angular momentum and, in the centrally symmetric case, the total energy.

1.1 Angular momentum and Kepler's second law

Definition 1.1 Let $\mathbf{r}\colon I \to \mathbb{R}^3$ be a C^1-map. The **angular momentum** of $\mathbf{r}$ about $\mathbf{0} \in \mathbb{R}^3$ is

$$\boxed{\mathbf{c}(t) := \mathbf{r}(t) \times \dot{\mathbf{r}}(t),}$$

where $\times$ denotes the usual cross product in $\mathbb{R}^3$.

Remark 1.2 The physical angular momentum of a body of mass m moving along a trajectory described by the map $\mathbf{r}$ is $m\mathbf{r} \times \dot{\mathbf{r}}$, so – strictly speaking – our angular momentum is the angular momentum per unit mass.

Proposition 1.3 *In the central force problem, the angular momentum is a constant of motion. For* $\mathbf{c} \neq \mathbf{0}$*, the motion takes place in the plane through* $\mathbf{0}$ *orthogonal to* $\mathbf{c}$*. For* $\mathbf{c} = \mathbf{0}$*, the motion is along a straight line through* $\mathbf{0}$*.*

Proof With (CFP) we find

$$\dot{\mathbf{c}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}.$$

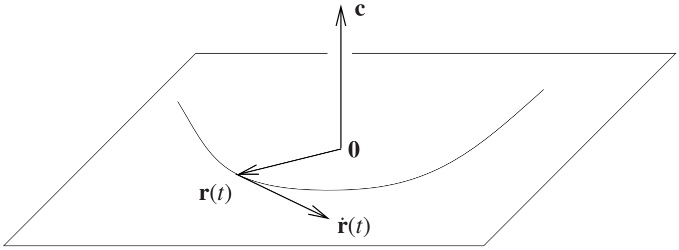

Figure 1.1 The angular momentum **c**.

This proves the first statement. Moreover, we have $\langle \mathbf{c}, \mathbf{r} \rangle = 0$ by the definition of $\mathbf{c}$. This settles the case $\mathbf{c} \neq \mathbf{0}$, see Figure 1.1.

In order to deal with the case $\mathbf{c} = \mathbf{0}$, we first derive a general identity in vector analysis. Consider a C^1-map $\mathbf{u}\colon I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Then

$$\dot{u} = \frac{\mathrm{d}}{\mathrm{d}t} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \frac{\langle \mathbf{u}, \dot{\mathbf{u}} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}} = \frac{\langle \mathbf{u}, \dot{\mathbf{u}} \rangle}{u},$$

hence

$$u\dot{u} = \langle \mathbf{u}, \dot{\mathbf{u}} \rangle.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{u}}{u}\right) = \frac{\dot{\mathbf{u}}u - \mathbf{u}\dot{u}}{u^2} = \frac{\dot{\mathbf{u}}\langle \mathbf{u}, \mathbf{u} \rangle - \mathbf{u}\langle \mathbf{u}, \dot{\mathbf{u}} \rangle}{u^3}.$$

With the **Graßmann identity**

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \langle \mathbf{a}, \mathbf{c} \rangle\, \mathbf{b} - \langle \mathbf{b}, \mathbf{c} \rangle\, \mathbf{a} \tag{1.1}$$

for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ (see Exercise 1.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{u}}{u}\right) = \frac{(\mathbf{u} \times \dot{\mathbf{u}}) \times \mathbf{u}}{u^3}.$$

Specialising to $\mathbf{u} = \mathbf{r}$, we get

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\mathbf{c} \times \mathbf{r}}{r^3}.} \tag{1.2}$$

Hence, for $\mathbf{c} = \mathbf{0}$ the vector $\mathbf{r}/r$ is constant, and $\mathbf{r}(t) = r(t) \cdot \mathbf{r}(t)/r(t)$ is always a positive multiple of this constant vector. □

Given a solution to (CFP), we may choose our coordinate system in such a way that the motion takes place in the xy-plane. Therefore, for the remainder of this section, we restrict our attention to planar curves

$$\alpha\colon I \to \mathbb{R}^2 \setminus \{\mathbf{0}\}.$$

We write

$$\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t)) = r(t) \cdot (\cos\theta(t), \sin\theta(t))$$

with $r(t) > 0$.

Remark 1.4 The transformation from polar to cartesian coordinates is described by the smooth map

$$\begin{array}{rccc} p\colon & \mathbb{R}^+ \times \mathbb{R} & \longrightarrow & \mathbb{R}^2 \setminus \{\mathbf{0}\} \\ & (r,\theta) & \longmapsto & (r\cos\theta, r\sin\theta). \end{array}$$

The Jacobian determinant of this map is

$$\det J_{p,(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0,$$

so p is a local diffeomorphism by the inverse function theorem. Moreover, p is a **covering map**, which means the following. For any point in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ one can find an open, path-connected neighbourhood U whose preimage $p^{-1}(U)$ is a non-empty disjoint union of sets U_λ, $\lambda \in \Lambda$, such that $p|_{U_\lambda}\colon U_\lambda \to U$ is a homeomorphism for each λ in the relevant index set Λ. (You are asked to verify this property in Exercise 1.3.)

The covering space property guarantees that any continuous curve $\boldsymbol{\alpha}\colon I \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ can be lifted to a continuous curve $\tilde{\boldsymbol{\alpha}}\colon I \to \mathbb{R}^+ \times \mathbb{R}$ with $p \circ \tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$, and this lift is uniquely determined by the choice of $\tilde{\boldsymbol{\alpha}}(t_0) \in p^{-1}(\boldsymbol{\alpha}(t_0))$ for some $t_0 \in I$. The local diffeomorphism property implies that if $\boldsymbol{\alpha}$ was of class C^k, so will be $\tilde{\boldsymbol{\alpha}}$.

The upshot is that a planar C^k-curve $\boldsymbol{\alpha}$ can be written in polar coordinates with C^k-functions r and θ. Of course, r is uniquely determined by $r = |\boldsymbol{\alpha}|$; the function θ is uniquely determined up to adding an integer multiple of 2π by an appropriate choice $\theta(t_0)$ at some $t_0 \in I$, and the requirement that θ at least be continuous.

In Exercise 1.4 you are asked to arrive at the same conclusion by an argument that does not involve any topological reasoning, but only the Picard–Lindelöf theorem.

Proposition 1.5 *Let*

$$\boldsymbol{\alpha} = r(\cos\theta, \sin\theta)\colon\ [t_0, t_1] \longrightarrow \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

be a C^1-curve with **angular velocity** $\dot{\theta} > 0$ *on* $[t_0, t_1]$ *and* $\theta(t_1) - \theta(t_0) < 2\pi$. *Then the area of*

$$D := \{s\boldsymbol{\alpha}(t)\colon\ t \in [t_0, t_1],\ s \in [0,1]\}$$

(see Figure 1.2) is given by

$$\operatorname{area}(D) = \frac{1}{2}\int_{t_0}^{t_1} r^2(t)\,\dot\theta(t)\,\mathrm{d}t.$$

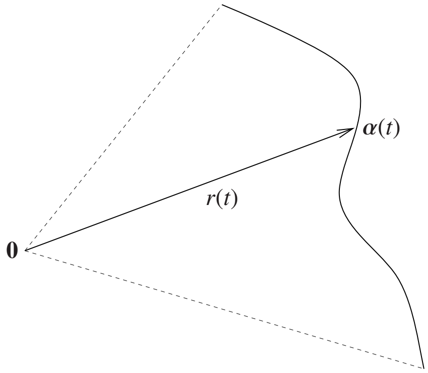

Figure 1.2 The area swept out by the position vector.

Proof Because of $\dot\theta > 0$, the inverse function theorem allows us to regard t and hence r as a function of θ. The area element $\mathrm{d}A$ is given in polar coordinates by $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$; this follows from the transformation formula:

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}\,\mathrm{d}r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta,$$

cf. Remark 1.4. Write $\theta_i = \theta(t_i)$, $i = 0, 1$. Then

$$\operatorname{area}(D) = \int_{\theta_0}^{\theta_1}\int_0^{r(\theta)} \rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \frac{1}{2}\int_{\theta_0}^{\theta_1} r^2(\theta)\,\mathrm{d}\theta.$$

The transformation rule (with $r(t) := r(\theta(t))$ and $\mathrm{d}\theta = \dot\theta\,\mathrm{d}t$) yields the claimed area formula. □

Theorem 1.6 (Kepler's second law) *The radial vector describing a solution of the central force problem sweeps out equal areas in equal intervals of time.*

Proof Choose coordinates such that the motion takes place in the xy-plane and $\mathbf{c}$ points in the positive z-direction. We may then write

$$\mathbf{r}(t) = r(t)\cdot(\cos\theta(t), \sin\theta(t), 0).$$

Hence

$$\dot{\mathbf{r}}(t) = (\dot r\cos\theta - r\dot\theta\sin\theta, \dot r\sin\theta + r\dot\theta\cos\theta, 0).$$

By Proposition 1.3, the angular momentum

$$\mathbf{c} = \mathbf{r} \times \dot{\mathbf{r}} = (0, 0, r^2\dot{\theta})$$

is constant. With our assumption on the orientation of $\mathbf{c}$ this implies $r^2\dot{\theta} = c$. (In particular, we have $\dot{\theta} \geq 0$, i.e. the motion is along a line or counterclockwise in the xy-plane.) Thus, according to Proposition 1.5, the area swept out during the time interval $[t_0, t_1]$ equals $c(t_1 - t_0)/2$. □

Remark 1.7 The converse to this theorem is also true: if a planar motion in a force field satisfies Kepler's second law with respect to the centre $\mathbf{0}$, the force field is central. This can be seen as follows. If Kepler's second law holds, then $r^2\dot{\theta}$ is constant. By the computation in the proof above, this means that $\mathbf{c}$ is constant. Hence $\mathbf{0} = \dot{\mathbf{c}} = \mathbf{r} \times \ddot{\mathbf{r}}$, which means that $\ddot{\mathbf{r}}$ is parallel to $\mathbf{r}$.

1.2 Conservation of energy

The motion of a particle in an open subset $\Omega \subset \mathbb{R}^3$ under the influence of a force field $\mathbf{F}\colon\ \Omega \to \mathbb{R}^3$ is described by the Newtonian differential equation

$$m\ddot{\mathbf{r}} = \mathbf{F}. \tag{N}$$

The force field $\mathbf{F}$ is called **conservative** if it has a **potential** $V\colon\ \Omega \to \mathbb{R}$, i.e. a C^1-function such that

$$\mathbf{F} = -\operatorname{grad} V.$$

Proposition 1.8 *For motions in a conservative force field, i.e. solutions of the Newtonian differential equation*

$$m\ddot{\mathbf{r}} = -\operatorname{grad} V(\mathbf{r}), \tag{N$_c$}$$

the **total energy**

$$E(t) := \frac{1}{2}mv^2(t) + V(\mathbf{r}(t))$$

is constant.

Proof With $v^2 = \langle \dot{\mathbf{r}}, \dot{\mathbf{r}} \rangle$ we see that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\langle \ddot{\mathbf{r}}, \dot{\mathbf{r}} \rangle + \langle \operatorname{grad} V(\mathbf{r}), \dot{\mathbf{r}} \rangle = \langle m\ddot{\mathbf{r}} + \operatorname{grad} V(\mathbf{r}), \dot{\mathbf{r}} \rangle = 0. \qquad □$$

Remark 1.9 The **kinetic energy** $mv^2/2$ is the work required to accelerate a body of mass m from rest to velocity $\mathbf{v}$. Indeed, if we accelerate the body along a path $\boldsymbol{\gamma}$ during the time interval $[0, T]$ from rest to the final velocity $\mathbf{v}(T)$, the work done is

$$\int_{\boldsymbol{\gamma}} \langle \mathbf{F}, \mathrm{d}\mathbf{s} \rangle = \int_0^T m \left\langle \mathbf{a}, \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} \right\rangle \mathrm{d}t = \int_0^T m \langle \dot{\mathbf{v}}, \mathbf{v} \rangle \, \mathrm{d}t = \frac{1}{2} m |\mathbf{v}(T)|^2.$$

Example 1.10 The force field describing the *centrally symmetric* central force problem is conservative. In order to see this, notice that in the centrally symmetric case the function f in (CFP) depends only on r rather than $\mathbf{r}$, i.e. we have

$$\mathbf{F} = -m f(r) \cdot \frac{\mathbf{r}}{r}.$$

I claim that the potential of this force field is given by the centrally symmetric function

$$V(r) := m \int_{r_0}^{r} f(\rho) \, \mathrm{d}\rho.$$

Indeed, we compute

$$\operatorname{grad} V(r) = V'(r) \cdot \operatorname{grad} r = m f(r) \cdot \frac{\mathbf{r}}{r}.$$

Remark 1.11 In the Newtonian case, with $f(r) = \mu / r^2$, the usual normalisation convention is to take $r_0 = \infty$, i.e.

$$V(r) = m \int_{\infty}^{r} \frac{\mu}{\rho^2} \, \mathrm{d}\rho = -\frac{m\mu}{r}.$$

Notes and references

The path-lifting property for coverings alluded to in Remark 1.4 is not difficult to show, but it requires a careful argument; see (Jänich, 2005, Chapter 9) or (McCleary, 2006, Chapter 8).

Kepler's second law was originally proved by Newton (1687) in Book 1, Proposition 1 of his *Principia*. A very useful guide to Newton's masterpiece is (Chandrasekhar, 1995).

Exercises

1.1 Let $\mathbb{R} \ni t \mapsto \mathbf{r}(t) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ be a solution of the central force problem (CFP). In this exercise we investigate the symmetries of such a solution.

(a) Show that $t \mapsto \mathbf{r}(t+b)$ for $b \in \mathbb{R}$ and $t \mapsto \mathbf{r}(-t)$ are likewise solutions of (CFP), i.e. we have invariance under time translation and time reversal.

(b) Suppose we are in the centrally symmetric situation, i.e. the function f in (CFP) depends only on r rather than $\mathbf{r}$. Show that in this case we have invariance under **isometries**, i.e. distance-preserving maps: if $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry with $A(\mathbf{0}) = \mathbf{0}$, so that A may be regarded as an element of the orthogonal group O(3), then $t \mapsto A\mathbf{r}(t)$ is likewise a solution.

1.2 Verify the Graßmann identity (1.1).

Hint: Observe that both sides of the equation are linear in $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. It therefore suffices to check equality when $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are chosen from the three standard basis vectors of $\mathbb{R}^3$.

1.3 Verify that the map p in Remark 1.4 is a covering map in the sense described there. As index set one can take $\Lambda = \mathbb{Z}$. How does one have to choose the neighbourhood U of a given point in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$?

1.4 Let $I \subset \mathbb{R}$ be an interval and $\boldsymbol{\alpha}\colon I \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ a C^k-curve for some $k \in \mathbb{N}$. We write

$$\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t)) = r(t)\big(\cos\theta(t), \sin\theta(t)\big)$$

with $r(t) > 0$. For every $t \in I$, the angle $\theta(t)$ is determined by $\boldsymbol{\alpha}(t)$ up to adding integer multiples of 2π. The aim of this exercise is to give an alternative proof of the observation in Remark 1.4, i.e. that θ can be chosen as a C^k-function.

First of all, we observe that r is determined by $r(t) = |\boldsymbol{\alpha}(t)|$ and hence is of class C^k, since $\boldsymbol{\alpha} \neq \mathbf{0}$. Thus, by passing to the curve $\boldsymbol{\alpha}/r$ we may assume without loss of generality that $r = 1$.

(a) (Uniqueness) Suppose the planar curve $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ has been written as $\boldsymbol{\alpha} = (\cos\theta, \sin\theta)$ with a C^k-function θ. Let $t_0 \in I$ and $\theta_0 = \theta(t_0)$. Show that

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \big(\alpha_1(s)\dot{\alpha}_2(s) - \dot{\alpha}_1(s)\alpha_2(s)\big)\,\mathrm{d}s.$$

(b) (Existence) Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $t_0 \in I$ be given. Choose θ_0 such that $\boldsymbol{\alpha}(t_0) = (\cos\theta_0, \sin\theta_0)$, and define θ via the equation in (a). Set $(\beta_1, \beta_2) := (\cos\theta, \sin\theta)$. Show that (α_1, α_2) and (β_1, β_2) are solutions of one and the same linear system of differential equations, with equal initial values at $t = t_0$. Conclude with the uniqueness statement in the Picard–Lindelöf theorem that $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)$. Verify that the assumptions of that theorem are indeed satisfied.

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1.5 In this exercise we want to give an alternative proof of the area formula for planar sets D of the form

$$D = \{s\boldsymbol{\alpha}(t)\colon\ t \in [t_0, t_1],\ s \in [0, 1]\},$$

where $\boldsymbol{\alpha} = r(\cos\theta, \sin\theta)\colon\ [t_0, t_1] \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is a C^1-curve with $\dot{\theta} > 0$ on $[t_0, t_1]$ and $\theta(t_1) - \theta(t_0) < 2\pi$, cf. Proposition 1.5.

(a) Show that the exterior normal vector $\mathbf{n}(t)$ to D in the boundary point $\boldsymbol{\alpha}(t) \in \partial D$ is given by

$$\mathbf{n}(t) = \frac{(\dot{\alpha}_2(t), -\dot{\alpha}_1(t))}{|\dot{\boldsymbol{\alpha}}(t)|}.$$

(b) Apply the divergence theorem (a.k.a. Gauß's theorem)

$$\int_D \operatorname{div} \mathbf{v}\, \mathrm{d}x\, \mathrm{d}y = \int_{\partial D} \langle \mathbf{v}, \mathbf{n} \rangle\, \mathrm{d}s$$

to the vector field $\mathbf{v}(x, y) = (x, y)$ in order to derive the area formula from Proposition 1.5. What is the contribution of the line segments in ∂D to the boundary integral?

1.6 (a) Let $t \mapsto \mathbf{r}(t) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ be a C^2-map. Set $\mathbf{v} := \dot{\mathbf{r}}$ and $\mathbf{c} := \mathbf{r} \times \dot{\mathbf{r}}$. Show that

$$v^2 = \dot{r}^2 + \frac{c^2}{r^2}.$$

(b) Now assume that $\mathbf{r}$ is a solution of the central force problem. So we may take $\mathbf{r}$ to be a planar curve and write $\mathbf{r} = r(\cos\theta, \sin\theta)$. Set $\mathbf{a} = \ddot{\mathbf{r}}$. Show that

$$a = \left| \frac{c^2}{r^3} - \ddot{r} \right|.$$

1.7 In this exercise we wish to derive a special case of Kepler's third law, see Theorem 3.7. Let $a \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$. Show that

$$\mathbf{r}(t) := a(\cos\omega t, \sin\omega t)$$

is a solution of the central force problem $\ddot{\mathbf{r}} = -\mathbf{r}/r^3$ (corresponding to the Newtonian law of gravitation $F \propto r^{-2}$) if and only if $|\omega| = 1/a^{3/2}$. What is the relation between the minimal period p (i.e. the time for one full rotation) and the radius a?

1.8 Verify the formula $\operatorname{grad} r = \mathbf{r}/r$ used in Example 1.10, and interpret this formula geometrically.

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2

Conic sections

Wenn sie erst den Kegel sieht, *muß sie glücklich sein*.

Thomas Bernhard, *Korrektur*

As we shall see in the next chapter, when we specialise in the central force problem to the Newtonian law of gravitational attraction, the solution curve will be an ellipse, a parabola, or a hyperbola. In the present chapter I give a bare bones introduction to the theory of these planar curves.

2.1 Ellipses

We take the following *gardener's construction* as the definition of an ellipse, see Figure 2.1, and then derive five other equivalent characterisations. The distance between two points $P, Q \in \mathbb{R}^2$ will be denoted by $|PQ|$.

Definition 2.1 Let F_1, F_2 be two points in $\mathbb{R}^2$ (possibly $F_1 = F_2$), and choose a real number $a > \frac{1}{2}|F_1F_2|$. The **ellipse** with **foci** F_1, F_2 and **semi-major axis** a is the set

$$\mathcal{E} := \{P \in \mathbb{R}^2 \colon\ |PF_1| + |PF_2| = 2a\}.$$

If you wish to lay out an elliptic flower bed in your garden, proceed as follows. Drive pegs into the ground at F_1 and F_2, take a piece of string of length greater than $|F_1F_2|$, tie one end each to the pegs, and draw an ellipse by moving a marker P around the two pegs while keeping the string stretched tight with the marker.

The midpoint Z of the two foci is called the **centre** of the ellipse. If $F_1 = F_2 = Z$, then $\mathcal{E}$ is a circle of radius a about Z. If $F_1 \neq F_2$, let P_1 be the point on