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Quantum Mechanics and Experience

Quantum mechanics is an extremely successful physical theory due to its accurate empirical predictions. The core of the theory, which is contained in various non-relativistic quantum theories, is the Schrödinger equation and the Born rule.[1] The Schrödinger equation governs the time evolution of the wave function assigned to a physical system, and the Born rule connects the wave function with the probabilities of possible results of a measurement on the system. In this chapter, I will introduce the core of quantum mechanics, especially the connections of its mathematical formalism with experience. The introduction is intended not to be complete but enough for the later analysis of the meaning of the wave function and the ontological content of quantum mechanics.

1.1 The Mathematical Formalism

The mathematical formalism of quantum mechanics is mainly composed of two parts. The first part assigns a mathematical object, the so-called wave function or quantum state, to a physical system appropriately prepared at a given instant.[2] The second part specifies how the wave function evolves with time. The evolution of the wave function is governed by the Schrödinger equation, whose concrete form is determined by the properties of the system and its interactions with environment.

There are two common representations for the wave function: the Hilbert space representation and the configuration space representation, which have their respective advantages. According to the Hilbert space representation, the wave function

[1] An apparent exception is collapse theories (Ghirardi, 2016). In these theories, however, the additional collapse term in the revised Schrödinger equation is so tiny for microscopic systems that it can be ignored in analyzing the ontological status and meaning of the wave function.

[2] It is worth noting that although all quantum theories assign the same wave function to an isolated physical system, different quantum theories, such as no-collapse theories and collapse theories, may assign different wave functions to a nonisolated physical system. The assignment, which depends on the concrete laws of motion in the theory, does not influence the ontological status and meaning of the wave function.

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is an unit vector or state vector in a Hilbert space, usually denoted by $|\psi(t)\rangle$ with Dirac's bracket notation. The Hilbert space is a complete vector space with scalar product, and its dimension and structure depend on the particular system. For example, the Hilbert space associated with a composite system is the tensor product of the Hilbert spaces associated with the systems of which it is composed.[3]

This structure of the Hilbert space can be seen more clearly from the configuration space representation. The configuration space of an N-body quantum system has $3N$ dimensions, and each point in the space can be specified by an ordered $3N$-tuple, where each group of three coordinates are position coordinates of each subsystem in three-dimensional space. The wave function of the system is a complex function on this configuration space,[4] and it can be written as $\psi(x_1, y_1, z_1, \ldots, x_N, y_N, z_N, t)$, where x_i, y_i, z_i are coordinates of the ith subsystem in the $3N$-dimensional configuration space. Moreover, the wave function is normalized; namely, the integral of the modulus squared of the wave function over the whole space is one. When the N subsystems are independent, the whole wave function can be decomposed as the product of the wave functions of the N subsystems, each of which lives in three-dimensional space.

For an N-body quantum system, there are also a $3N$-dimensional space and wave functions on the space for other properties of the system besides position. For example, the momentum space of an N-body system is a $3N$-dimensional space parameterized by $3N$ momentum coordinates, and the momentum wave function is a complex function on this space. Here the Hilbert space representation is more convenient. Every measurable property or observable of a physical system is represented by a Hermitian operator on the Hilbert space associated with the system, and the wave functions for different properties such as position and momentum may be transformed into each other by considering the relationship between the corresponding operators of these properties.

The second part of the mathematical formalism of quantum mechanics specifies how the wave function assigned to a physical system evolves with time. The time evolution of the wave function, $|\psi(t)\rangle$, is governed by the Schrödinger equation

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle, \tag{1.1}$$

where $\hbar$ is Planck's constant divided by 2π, and H is the Hamiltonian operator that depends on the energy properties of the system. The time evolution is linear and unitary in the sense that the Hamiltonian is independent of the evolving wave

[3] Similarly, the Hilbert space associated with independent properties is the tensor product of the Hilbert spaces associated with each property.
[4] To be consistent with convention, I will also say "the wave function of a physical system," but it still means "the wave function assigned to a physical system."

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function and it keeps the normalization of the wave function unchanged. The concrete forms of the Hamiltonian and the Schrödinger equation depend on the studied system and its interactions with other systems in the environment. For example, the wave function of an electron evolving in an external potential obeys the following Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,y,z,t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+V(x,y,z,t)\right]\psi(x,y,z,t), \tag{1.2}$$

where $\psi(x,y,z,t)$ is the wave function of the electron, m is the mass of the electron, and $V(x,y,z,t)$ is the external potential.

1.2 The Born Rule

What is the empirical content of quantum mechanics? Or how does the wave function assigned to a physical system relate to the results of measurements on the system? The well-known connection rule is the Born rule, which has been precisely tested by experiments. It says that a (projective) measurement of an observable A on a system with the wave function $|\psi\rangle$ will randomly obtain one of the eigenvalues of A, and the probability of obtaining an eigenvalue a_i is given by $|\langle a_i|\psi\rangle|^2$, where $|a_i\rangle$ is the eigenstate corresponding to the eigenvalue a_i.

The Born rule can also be formulated in the language of configuration space. It says that the integral of the modulus squared of the wave function over a certain region of the configuration space associated with a property of a physical system gives the probability of the measurement of the property of the system obtaining the values inside the region. For example, for a physical system whose wave function is $\psi(x,y,z,t)$, $|\psi(x,y,z,t)|^2dxdydz$ represents the probability of a position measurement on the system obtaining a result between (x,y,z) and $(x+dx,y+dy,z+dz)$, and $|\psi(x,y,z,t)|^2$ is the corresponding probability density in position (x,y,z). Similarly, for an N-body system whose wave function is $\psi(x_1,y_1,z_1,\ldots,x_N,y_N,z_N,t)$, $|\psi(x_1,y_1,z_1,\ldots,x_N,y_N,z_N,t)|^2$ represents the probability density that a position measurement on the first subsystem obtains result (x_1,y_1,z_1), and a position measurement on the second subsystem obtains result $(x_2,y_2,z_2),\ldots$, and a position measurement on the Nth subsystem obtains result (x_N,y_N,z_N).

The Born rule provides a probabilistic connection between the wave function and the results of measurements. However, it may be not the only connection rule, as the involved measurements are only one kind of measurements, projective measurements. In order to know whether there are other possible connections between quantum mechanics and experience, we need to analyze measurements in more detail.

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A measurement is an interaction between a measured system and a measuring device. It can be described by using the standard von Neumann procedure. Suppose the wave function of the measured system is $|\psi\rangle$ at a given instant $t = 0$, and the initial wave function of the pointer of a measuring device at $t = 0$ is a Gaussian wavepacket of very small width w_0 centered in initial position x_0, denoted by $|\phi(x_0)\rangle$. The total Hamiltonian of the combined system can be written as

$$H = H_S + H_D + H_I, \tag{1.3}$$

where H_S and H_D are the free Hamiltonians of the measured system and the measuring device, respectively, and H_I is the interaction Hamiltonian coupling the measured system to the measuring device, which can be further written as

$$H_I = g(t)PA, \tag{1.4}$$

where P is the momentum of the pointer of the measuring device, A is the measured observable, and $g(t)$ represents the time-dependent coupling strength of the interaction, which is a smooth function normalized to $\int dtg(t) = 1$ during the measurement interval τ, and $g(0) = g(\tau) = 0$.

It has been known that there are different types of measurements, depending on the interaction strength and time and whether the measured system is appropriately protected, and so on. The most common measurements are projective measurements involved in the Born rule. For a projective measurement, the interaction H_I is of very short duration and so strong that it dominates the rest of the Hamiltonian, and thus the effect of the free Hamiltonians of the measuring device and the measured system can be neglected. Then the state of the combined system at the end of the interaction can be written as

$$|t = \tau\rangle = e^{-\frac{i}{\hbar}PA} |\psi\rangle |\phi(x_0)\rangle . \tag{1.5}$$

By expanding $|\psi\rangle$ in the eigenstates of A, $|a_i\rangle$, we obtain

$$|t = \tau\rangle = \sum_i e^{-\frac{i}{\hbar}Pa_i} c_i |a_i\rangle |\phi(x_0)\rangle , \tag{1.6}$$

where c_i are the expansion coefficients. The exponential term shifts the center of the pointer by a_i:

$$|t = \tau\rangle = \sum_i c_i |a_i\rangle |\phi(x_0 + a_i)\rangle . \tag{1.7}$$

This is an entangled state, where the eigenstates of A with eigenvalues a_i are correlated to the measuring device states in which the pointer is shifted by these eigenvalues a_i.

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The Born rule tells us (and we also know by experience) that the result of this projective measurement is one of the eigenvalues of the measured observable, say, a_i, with probability $|c_i|^2$. However, we still don't know whether this entangled superposition is the final state of the combining system after the measurement.[5] The appearance of the definite result seems apparently incompatible with the superposed state. This is the notorious measurement problem. I will try to solve this problem in Chapter 8.

1.3 A Definite Connection with Experience

It is not surprising that since the interaction between the measured system and the measuring device is very strong during a projective measurement, the measurement disturbs the measured system and changes its wave function greatly. This is not a good measurement. A good measurement is required not to disturb the state of the measured system so that it can measure the realistic properties of the system. This is possible for projective measurements only when the initial state of the measured system is an eigenstate of the measured observable. In this case, the final state of the combining system is not an entangled state but a product state, such as:

$$|t = \tau\rangle = |a_i\rangle \, |\phi(x_0 + a_i)\rangle \, . \tag{1.8}$$

According to the Born rule, this projective measurement obtains a definite result a_i.

A general way to make a good measurement is to protect the measured state from being changed when the measurement is being made. A universal protection scheme is via the quantum Zeno effect (Aharonov, Anandan, and Vaidman, 1993).[6] Let us see how this can be done. We make projective measurements of an observable O, of which the measured state $|\psi\rangle$ is a nondegenerate eigenstate, a large number of times that are dense in a very short measurement interval $[0, \tau]$. For example, O is measured in $[0, \tau]$ at times $t_n = (n/N)\tau, n = 1, 2, \ldots, N$, where N is an arbitrarily large number. At the same time, we make the same projective measurement of an observable A in the interval $[0, \tau]$ as in the last section, which is described by the interaction Hamiltonian (1.4).

[5] In other words, it is still unknown how the wave function evolves during a projective measurement. In standard quantum mechanics, which is formulated by Dirac (1930) and von Neumann (1932), it is assumed that after a projective measurement of an observable, the entangled superposition formed by the Schrödinger evolution collapses to one of the eigenstates of the observable that corresponds to the result of the measurement. This assumption is called the collapse postulate. For a helpful introduction of standard quantum mechanics for philosophers, see Ismael (2015).

[6] Another protection scheme is to introduce a protective potential such that the measured wave function of a quantum system is a nondegenerate energy eigenstate of the Hamiltonian of the system with finite gap to neighboring energy eigenstates (Aharonov and Vaidman, 1993). By this scheme, the measurement of an observable is required to be weak and adiabatic.

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As noted before, since the interaction H_I is of very short duration and so strong that it dominates the rest of the Hamiltonian, the effect of the free Hamiltonians of the measuring device and the measured system can be neglected. Then the branch of the state of the combined system after τ, in which each projective measurement of O results in the state of the measured system being in $|\psi\rangle$, is given by

$$\begin{aligned} |t=\tau\rangle &= |\psi\rangle\langle\psi|e^{-\frac{i}{\hbar}\frac{\tau}{N}H(t_N)}\ldots|\psi\rangle\langle\psi|e^{-\frac{i}{\hbar}\frac{\tau}{N}H(t_2)}|\psi\rangle\langle\psi| \\ &\quad\times e^{-\frac{i}{\hbar}\frac{\tau}{N}H(t_1)}|\psi\rangle|\phi(x_0)\rangle \\ &= |\psi\rangle\langle\psi|e^{-\frac{i}{\hbar}\frac{\tau}{N}g(t_N)PA}\ldots|\psi\rangle\langle\psi|e^{-\frac{i}{\hbar}\frac{\tau}{N}g(t_2)PA}|\psi\rangle\langle\psi| \\ &\quad\times e^{-\frac{i}{\hbar}\frac{\tau}{N}g(t_1)PA}|\psi\rangle|\phi(x_0)\rangle, \end{aligned} \tag{1.9}$$

where $|\phi(x_0)\rangle$ is the initial wave function of the pointer of the measuring device, which is supposed to be a Gaussian wavepacket of very small width centered in initial position x_0.

Thus in the limit of $N\to\infty$, we have

$$|t=\tau\rangle = |\psi\rangle\, e^{-\frac{i}{\hbar}\int_0^\tau g(t)\langle\psi|A|\psi\rangle P dt}|\phi(x_0)\rangle = |\psi\rangle|\phi(x_0+\langle A\rangle)\rangle, \tag{1.10}$$

where $\langle A\rangle \equiv \langle\psi|A|\psi\rangle$ is the expectation value of A in the measured state $|\psi\rangle$. Since the modulus squared of the amplitude of this branch approaches one when $N\to\infty$, this state will be the state of the combined system after τ.[7] It can be seen that after the measurement, the measuring device state and the system state are not entangled, and the pointer of the measuring device is shifted by the expectation value $\langle A\rangle$.[8]

This demonstrates that for an arbitrary state of a quantum system at a given instant, we can protect the state from being changed via the quantum Zeno effect, and a projective measurement of an observable, which is made at the same time, yields a definite measurement result, the expectation value of the observable in the measured state. Such measurements have been called protective measurements (Aharonov and Vaidman, 1993; Aharonov, Anandan, and Vaidman, 1993; Vaidman, 2009).

In fact, it can be shown that if the measured state is not changed during a projective measurement, then the result must be the expectation value of the measured

[7] It is worth noting that the possible collapse of the wave function resulting from the projective measurements of O does not influence this result. The reason is that the probability of the measured state collapsing to another state different from $|\psi\rangle$ after each projective measurement of O is proportional to $1/N^2$, and thus the sum of these probabilities is proportional to $1/N$ after τ and approaches zero when $N\to\infty$. Moreover, since the pointer of a measuring device may be a microscopic system, whose shift can be further read out by another measuring device, the effect of the possible collapse of the wave function resulting from the projective measurements of A can also be ignored.

[8] Note that after the measurement the pointer wavepacket does not spread, and the width of the wavepacket is the same as the initial width. This ensures that the pointer shift can represent a valid measurement result.

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observable in the measured state. In this case, the evolution of the state of the combined system is

$$|\psi(0)\rangle\,|\phi(0)\rangle \rightarrow |\psi(t)\rangle\,|\phi(t)\rangle\,, \quad t > 0, \tag{1.11}$$

where $|\phi(0)\rangle$ and $|\phi(t)\rangle$ are the states of the measuring device at instants 0 and t, respectively; $|\psi(0)\rangle$ and $|\psi(t)\rangle$ are the states of the measured system at instants 0 and t, respectively; and $|\psi(t)\rangle$ is the same as $|\psi(0)\rangle$ up to a phase factor during the measurement interval $[0, \tau]$. The interaction Hamiltonian is still given by (1.4). Then by Ehrenfest's theorem we have

$$\frac{d}{dt}\langle\psi(t)\phi(t)|X\,|\psi(t)\phi(t)\rangle = g(t)\langle\psi(0)|A\,|\psi(0)\rangle\,, \tag{1.12}$$

where X is the pointer variable. This further leads to

$$\langle\phi(\tau)|X\,|\phi(\tau)\rangle - \langle\phi(0)|X\,|\phi(0)\rangle = \langle\psi(0)|A\,|\psi(0)\rangle\,, \tag{1.13}$$

which means that the shift of the center of the pointer of the measuring device is the expectation value of the measured observable in the measured state. This clearly demonstrates that the result of a measurement that does not disturb the measured state is the expectation value of the measured observable in the measured state.

Since the wave function can be reconstructed from the expectation values of a sufficient number of observables, the wave function of a single quantum system can be measured by a series of protective measurements. Let the explicit form of the measured state at a given instant t be $\psi(x)$, and the measured observable A be (normalized) projection operators on small spatial regions V_n having volume v_n:

$$A = \begin{cases} \frac{1}{v_n}, & \text{if } x \in V_n, \\ 0, & \text{if } x \notin V_n. \end{cases} \tag{1.14}$$

A protective measurement of A then yields

$$\langle A\rangle = \frac{1}{v_n}\int_{V_n} |\psi(x)|^2 dv, \tag{1.15}$$

which is the average of the density $\rho(x) = |\psi(x)|^2$ over the small region V_n. Similarly, we can measure another observable $B = \frac{\hbar}{2mi}(A\nabla + \nabla A)$. The measurement yields

$$\langle B\rangle = \frac{1}{v_n}\int_{V_n} \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)dv = \frac{1}{v_n}\int_{V_n} j(x)dv. \tag{1.16}$$

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This is the average value of the flux density $j(x)$ in the region V_n. Then when $v_n \to 0$ and after performing measurements in sufficiently many regions V_n we can measure $\rho(x)$ and $j(x)$ everywhere in space. Since the wave function $\psi(x, t)$ can be uniquely expressed by $\rho(x, t)$ and $j(x, t)$ (except for an overall phase factor), the whole wave function of the measured system at a given instant can be measured by protective measurements.

Protective measurements provide a definite, direct connection between the wave function assigned to a physical system and the results of measurements on the system, and the connection is determined only by the linear Schrödinger evolution.[9] As I will argue later in this book, although this connection seems less well known, it will be extremely important for understanding the meaning of the wave function and searching for the ontology of quantum mechanics.

[9] Note that besides the wave function there are also state-independent quantities such as m (mass) and Q (charge) in the Schrödinger equation, and the measurement of such a quantity will obtain a definite result. This is also a definite, direct connection between the mathematical formalism of quantum mechanics and experience.

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