CHAPTER 1

Basic Concepts of Random Sets

In this chapter we lay the foundation for our analysis of econometric models based on random set theory. In particular, we formally define random closed sets and their distributions and connect these concepts with the corresponding ones for random variables and random vectors.

1.1 RANDOM CLOSED SETS

Realizations of a Random Set

The first step in defining a random element is to describe the family of its values. For a random set, the values will be subsets of a certain carrier space $\mathfrak{X}$, which is often taken to be the Euclidean space $\mathbb{R}^d$, but may well be different, e.g., a cube in $\mathbb{R}^d$, a sphere, a general discrete set, or an infinite-dimensional space like the space of (say, continuous) functions. It is always assumed that $\mathfrak{X}$ has the structure of a topological space.

The family of *all* subsets of any reasonably rich space is immense, and it is impossible to define a non-trivial distribution on it. In view of this, one typically considers certain families of sets with particular topological properties, e.g., closed, compact, or open sets, or with some further properties, most importantly convex sets. The conventional theory of random sets deals with random *closed* sets. An advantage of this approach is that random points (or random sets that consist of a single point, also called singletons) are closed, and so the theory of random closed sets then includes the classical case of random points or random vectors.

Denote by $\mathcal{F}$ the *family of closed subsets* of the carrier space $\mathfrak{X}$ and by F a generic closed set. Recall that the empty set and the whole $\mathfrak{X}$ are closed and so belong to $\mathcal{F}$. The set F is *closed* if it contains the limit of each convergent sequence of its elements, i.e., if $x_n \in F$ and $x_n \to x$ as $n \to \infty$, then the limit x belongs to F. For a general set $A \subseteq \mathfrak{X}$, denote by $\mathrm{cl}(A)$ its closure – that is, the smallest closed set that contains A. The complement of each closed set is open.

In the following, fix a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$, where all random elements (and random sets) will be defined, so that Ω is the space of elementary events equipped with σ-algebra $\mathfrak{A}$ and probability measure $\mathbf{P}$. For simplicity, assume that the σ-algebra $\mathfrak{A}$ is complete – that is, for all $A \in \mathfrak{A}$ with $\mathbf{P}(A) = 0$ and all $A' \subset A$ one has $A' \in \mathfrak{A}$.

Measurability and Traps

A random closed set is a measurable map $\boldsymbol{X} : \Omega \mapsto \mathcal{F}$. Its measurability is defined by specifying the family of functionals of $\boldsymbol{X}$ that are random variables. A possible idea would be to require that the indicator function $\mathbf{1}_{u \in \boldsymbol{X}}$ (which equals 1 if $u \in \boldsymbol{X}$ and equals 0 otherwise) is a stochastic process, i.e., each of its values is a random variable. However, this does not work well for random sets $\boldsymbol{X}$ that are “thin,” e.g., for $\boldsymbol{X} = \{\boldsymbol{x}\}$ being a random singleton. For instance, if $\boldsymbol{x}$ is a point in the Euclidean space with an absolutely continuous distribution, then $\{u \in \boldsymbol{X}\} = \{u = \boldsymbol{x}\}$ has probability zero, so the measurability condition of the *indicator* function

$$\mathbf{1}_{u \in \boldsymbol{X}} = \mathbf{1}_{\boldsymbol{x}=u}$$

does not impose any extra requirement on $\boldsymbol{x}$ (given that the underlying σ-algebra $\mathfrak{A}$ is complete). The same problem arises if $\boldsymbol{X}$ is a segment or a curve in the plane.

Hence, a definition of measurability based on indicators of points is not suitable for the purpose of defining a random closed set. Note that too strict measurability conditions unnecessarily restrict the possible examples of random sets. On the other hand, too weak measurability conditions do not ensure that important functionals of a random set become random variables. The measurability of a random closed set is therefore defined by replacing $\mathbf{1}_{x \in \boldsymbol{X}}$ with the indicator of the event $\{\boldsymbol{X} \cap K \neq \emptyset\}$ for some test sets K. In other words, the aim is to “trap” a random set $\boldsymbol{X}$ using a trap given by K. If K is only a singleton, then such a trap can be too meager to catch a thin set, and so is replaced by larger traps, being general compact sets. Let $\mathcal{K}$ be the family of all compact subsets of $\mathfrak{X}$ and let K be a generic compact set. In $\mathfrak{X} = \mathbb{R}^d$, compact sets are the closed and bounded ones.

Definition 1.1 A map $\boldsymbol{X}$ from a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ to the family $\mathcal{F}$ of closed subsets of a locally compact second countable Hausdorff space $\mathfrak{X}$ is called a *random closed set* if

$$\boldsymbol{X}^{-}(K) = \{\omega \in \Omega : \ \boldsymbol{X}(\omega) \cap K \neq \emptyset\} \tag{1.1}$$

belongs to the σ-algebra $\mathfrak{A}$ on Ω for each compact set K in $\mathfrak{X}$.

Many spaces of interest, most importantly the Euclidean space $\mathbb{R}^d$ and discrete spaces, are locally compact second countable Hausdorff (see the chapter notes for a short mathematical explanation of this property). Unless

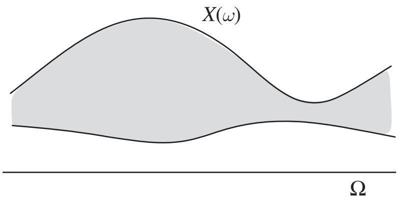

Figure 1.1 Random interval as function of $\omega \in \Omega$.

stated otherwise, the carrier space $\mathfrak{X}$ is assumed to be locally compact second countable Hausdorff. Random closed sets in more general spaces are considered in Theorem 2.10. In the following, we mostly assume that the carrier space $\mathfrak{X}$ is the Euclidean space $\mathbb{R}^d$ with the Euclidean metric $\mathbf{d}(\cdot,\cdot)$, norm $\|\cdot\|$, and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

Definition 1.1 yields that a random closed set is a measurable map from the given probability space to the family of closed sets equipped with the σ-algebra generated by the families of closed sets $\{F \in \mathcal{F} : F \cap K \neq \emptyset\}$ for all $K \in \mathcal{K}$, where $\mathcal{K}$ denotes the family of compact subsets of $\mathfrak{X}$. In what follows, ω is usually omitted, so that $\{\boldsymbol{X} \cap K \neq \emptyset\}$ is a shorthand for $\{\omega : \boldsymbol{X}(\omega) \cap K \neq \emptyset\}$. Random closed sets are usually denoted by bold capital letters $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$.

Figure 1.1 shows a random closed set (specifically, a random interval) as a function of ω. The function $\boldsymbol{X} : \Omega \mapsto \mathcal{F}$ is an example of a set-valued function, and $\boldsymbol{X}^-(K)$ is said to be the inverse of $\boldsymbol{X}$. In the same way it is possible to define measurability of any set-valued function with closed values that does not have to be defined on a probability space.

A random *compact* set is a random closed set that is compact with probability one, so that almost all values of $\boldsymbol{X}$ are compact sets. A random *convex* closed set is defined similarly, so that $\boldsymbol{X}(\omega)$ is a convex closed set for almost all ω. An almost surely non-empty random convex compact set in $\mathbb{R}^d$ is often called a *random convex body*.

Consider the random set $\boldsymbol{X} = [\boldsymbol{x}, \infty)$ on the real line (see Example 1.3) and take $K = \{t\}$ being a singleton. Then

$$\{\boldsymbol{X} \cap K \neq \emptyset\} = \{\boldsymbol{x} \leq t\}.$$

It is well known that for a random variable $\boldsymbol{x}$, the measurability of the events $\{\boldsymbol{x} \leq t\}$ for all t is equivalent to the measurability of the events $\{\boldsymbol{x} < t\}$ for all t. This is shown by approximation, namely

$$\{\boldsymbol{x} < t\} = \bigcup_{n=1}^{\infty} \left\{\boldsymbol{x} \leq t - n^{-1}\right\}.$$

A similar logic can be applied to random sets. The measurability of hitting events for compact sets can be extended to hitting events for open sets and even Borel sets. The ultimate property of such extension is the content of the Fundamental Measurability theorem (Theorem 2.10). Our topological assumptions on $\mathfrak{X}$ (that it is locally compact second countable Hausdorff) guarantee that any open set G can be approximated by compact sets, so that $K_n \uparrow G$ for a sequence of compact sets K_n, $n \geq 1$. Then

$$\{X \cap G \neq \emptyset\} = \bigcup_{n \geq 1} \{X \cap K_n \neq \emptyset\} \tag{1.2}$$

is a random event for each open set G. In one rather special case, the set G is the whole space, say $\mathbb{R}^d$, and $\{X \cap \mathbb{R}^d \neq \emptyset\} = \{X \neq \emptyset\}$ means that X is non-empty. Furthermore, for any closed set F,

$$\{X \subseteq F\} = \{X \cap F^c = \emptyset\}$$

is also a measurable event, since the complement F^c is an open set.

Examples of Random Sets Defined by Random Points

Example 1.2 (Random singleton) Random elements in $\mathfrak{X}$ are defined as measurable maps from $(\Omega, \mathfrak{A}, \mathbf{P})$ to the space $\mathfrak{X}$ equipped with its Borel σ-algebra $\mathcal{B}(\mathfrak{X})$. Then the singleton $X = \{x\}$ is a random closed set. Indeed,

$$\{X \cap K \neq \emptyset\} = \{x \in K\} \in \mathfrak{A}$$

for each compact set K.

Example 1.3 (Random half-line) If x is a random variable on the real line $\mathbb{R}$, then the *half-lines* $X = [x, \infty)$ and $Y = (-\infty, x]$ are random closed sets on $\mathfrak{X} = \mathbb{R}$. Indeed,

$$\{X \cap K \neq \emptyset\} = \{x \leq \sup K\} \in \mathfrak{A},$$
$$\{Y \cap K \neq \emptyset\} = \{x \geq \inf K\} \in \mathfrak{A},$$

for each compact set K. This example is useful for relating the classical notion of the cumulative distribution function of random variables to more general concepts arising in the theory of random sets.

Example 1.4 (Random ball) Let $\mathfrak{X}$ be equipped with a metric $\mathbf{d}$. A random *ball* $X = B_y(x)$ with center x and radius y is a random closed set if x is a random vector and y is a non-negative random variable. Then

$$\{X \cap K \neq \emptyset\} = \{y \geq \mathbf{d}(x, K)\},$$

where $\mathbf{d}(x, K)$ is the distance from x to the nearest point in K. Since both y and $\mathbf{d}(x, K)$ are random variables, it is immediately clear that $\{X \cap K \neq \emptyset\} \in \mathfrak{A}$. If the joint distribution of (x, y) depends on a certain parameter, we obtain a parametric family of distributions for random balls.

Example 1.5 (Finite random sets) The set of three points $X = \{x_1, x_2, x_3\}$ is a random closed set if x_1, x_2, x_3 are random elements in $\mathfrak{X}$. Indeed,

$$\{X \cap K \neq \emptyset\} = \bigcup_{i=1}^{3} \{x_i \in K\} \in \mathfrak{A}.$$

It is also possible to consider a random *finite* set X formed by an arbitrary number N of (possibly dependent) random elements in $\mathfrak{X}$. The cardinality N of X may be a random variable. In this case it is typical to call X a finite *point process*.

Example 1.6 (Random polytopes) A random triangle $Y = \Delta_{x_1,x_2,x_3}$ obtained as the convex hull of $X = \{x_1, x_2, x_3\}$ in $\mathbb{R}^d$ is also a random closed set. However, it is difficult to check directly that Y is measurable, since it is rather cumbersome to express the event $\{Y \cap K \neq \emptyset\}$ in terms of the vertices x_1, x_2, x_3. Measurability will be shown later by an application of the Fundamental Measurability theorem (Theorem 2.11 in the next chapter). Similarly, it is possible to consider random *polytopes* that appear as convex hulls of any (fixed or random) number of random points in the Euclidean space.

Example 1.7 (Finite carrier space) Let $\mathfrak{X}$ be a finite set that is equipped with the discrete metric meaning that all its subsets are closed and compact. Then X is a random closed set if and only if $\{u \in X\}$ is a random event for all $u \in \mathfrak{X}$.

Random Sets Related to Deterministic and Random Functions

Example 1.8 (Deterministic function at random level) Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a deterministic function, and let x be a random variable. If f is continuous, then $X = \{x : f(x) = x\}$ is a random closed set called the *level set* of f.

If f is *upper semicontinuous*, i.e.,

$$f(u) \geq \limsup_{v \to u} f(v) \tag{1.3}$$

for all x, then $Y = \{u : f(u) \geq x\}$ is closed and defines a random closed set (called the upper excursion set). Indeed,

$$\{Y \cap K \neq \emptyset\} = \left\{\sup_{u \in K} f(u) \geq x\right\} \in \mathfrak{A},$$

since x is a random variable. The distributions of X and Y are determined by the distribution of x and the choice of f. Both $X = f^{-1}(\{x\})$ and $Y = f^{-1}([x, \infty))$ can be obtained as inverse images. Note that f is called lower semicontinuous if $(-f)$ is upper semicontinuous.

Example 1.9 (Excursions of random functions) Let $x(t)$, $t \in \mathbb{R}$, be a real-valued stochastic process. If this process has continuous sample paths, then $\{t : x(t) = c\}$ is a random closed set for each $c \in \mathbb{R}$. For instance, if $x(t) = z_n t^n + \cdots + z_1 t + z_0$ is the polynomial of degree n in $t \in \mathbb{R}$ with random coefficients, then $X = \{t : x(t) = 0\}$ is the random set of its roots.

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If $\boldsymbol{x}$ has almost surely lower semicontinuous sample paths, then the *lower excursion* set $\boldsymbol{X} = \{t : \boldsymbol{x}(t) \le c\}$ and the *epigraph*

$$\boldsymbol{Y} = \operatorname{epi} \boldsymbol{x} = \{(t, s) \in \mathbb{R} \times \mathbb{R} : \boldsymbol{x}(t) \ge s\}$$

are random closed sets. For instance,

$$\{\boldsymbol{X} \cap K \ne \emptyset\} = \Big\{\inf_{t \in K} \boldsymbol{x}(t) \le c\Big\} \in \mathfrak{A}.$$

In view of this, statements about the supremum of a stochastic process can be formulated in terms of the corresponding excursion sets. The same construction works for random functions indexed by multidimensional arguments. Lower excursion sets appear as solutions to inequalities or systems of inequalities in partial identification problems (see Section 5.2).

Example 1.10 (Half-space) Let

$$\boldsymbol{X} = \{u \in \mathbb{R}^d : \boldsymbol{y}^\top u \le 1\}$$

be a random *half-space* determined by a random vector $\boldsymbol{y}$ in $\mathbb{R}^d$ – that is, $\boldsymbol{X}$ is the solution to a random linear inequality. We consider all vectors in $\mathbb{R}^d$ as columns, so that

$$\boldsymbol{y}^\top u = \sum_{i=1}^{d} \boldsymbol{y}_i u_i$$

denotes the scalar product of $\boldsymbol{y}$ and u. Note that the right-hand side of the inequality is set to 1 without loss of generality, since the scaling can be incorporated into $\boldsymbol{y}$. This definition of $\boldsymbol{X}$ may be considered a special case of Example 1.9 for the excursion set of a linear function. If A is a random matrix and $\boldsymbol{y}$ is a random vector, then the solution of the random linear equations $\{u \in \mathbb{R}^d : Au \le \boldsymbol{y}\}$ is a random closed set.

Examples of Random Sets in Partial Identification

Example 1.11 (Random interval) Interval data is a commonplace problem in economics and the social sciences more generally. Let $\boldsymbol{Y} = [\boldsymbol{y}_\mathrm{L}, \boldsymbol{y}_\mathrm{U}]$ be a *random interval* on $\mathbb{R}$, where $\boldsymbol{y}_\mathrm{L}$ and $\boldsymbol{y}_\mathrm{U}$ are two (dependent) random variables such that $\boldsymbol{y}_\mathrm{L} \le \boldsymbol{y}_\mathrm{U}$ almost surely. If $K = [a, b]$, then

$$\{\boldsymbol{Y} \cap K \ne \emptyset\} = \{\boldsymbol{y}_\mathrm{L} < a, \boldsymbol{y}_\mathrm{U} \ge a\} \cup \{\boldsymbol{y}_\mathrm{L} \in [a, b]\} \in \mathfrak{A},$$

because $\boldsymbol{y}_\mathrm{L}$ and $\boldsymbol{y}_\mathrm{U}$ are random variables. If $K \subset \mathbb{R}$ is an arbitrary compact set, then $\{\boldsymbol{Y} \cap K = \emptyset\}$ if and only if $\boldsymbol{Y}$ is a subset of the complement of K. The complement of K is the union of an at most countable number of disjoint open intervals, so it suffices to note that $\{\boldsymbol{Y} \subset (a, b)\} = \{a < \boldsymbol{y}_\mathrm{L} \le \boldsymbol{y}_\mathrm{U} < b\}$ is a measurable event.

Example 1.12 (Revealed preferences) Suppose that an individual chooses an action from a finite ordered choice set $D = \{d_1, d_2, d_3\}$, with $d_1 < d_2 < d_3$, to maximize her utility function, which for simplicity is assumed to equal

$$U(d_i) = -\boldsymbol{x}_i - \psi d_i,$$

where for $i = 1, 2, 3$, $\boldsymbol{x}_i$ is a random variable, observable by the researcher, which characterizes action d_i, and $\psi \in \Psi \subset \mathbb{R}$ is an individual-specific preference parameter with Ψ a compact set. Then the values of ψ consistent with the model and observed choices form a random closed set. To see this, note that, if, for example, the individual chooses action d_2, revealed preference arguments yield

$$\frac{\boldsymbol{x}_2 - \boldsymbol{x}_3}{d_3 - d_2} \le \psi \le \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{d_2 - d_1}.$$

A similar argument holds for the case that the individual chooses d_1 or d_3. Measurability follows from Example 1.11.

Example 1.13 (Treatment response) Consider a classic selection problem in which an individual may receive a treatment $t \in \{0, 1\}$ and let $\boldsymbol{y} : \{0, 1\} \mapsto \mathcal{Y}$ denote a (random) response function mapping treatments $t \in \{0, 1\}$ into outcomes $\boldsymbol{y}(t) \in \mathcal{Y}$, with $\mathcal{Y}$ a compact set in $\mathbb{R}$. Without loss of generality, assume that $\min \mathcal{Y} = 0$ and $\max \mathcal{Y} = 1$, so that $\mathcal{Y}$ contains both 0 and 1. Let $\boldsymbol{z} \in \{0, 1\}$ be a random variable denoting the treatment received by the individual. The researcher observes the tuple $(\boldsymbol{z}, \boldsymbol{y})$ of treatment received and outcome experienced by the individual and is interested in inference on functionals of the potential outcome $\boldsymbol{y}(t)$, e.g., its expectation $\mathbf{E}(\boldsymbol{y}(t))$ or the distribution $P_{\boldsymbol{y}(t)}$. For $\boldsymbol{z} = t$, the outcome $\boldsymbol{y}(t) = \boldsymbol{y}(\boldsymbol{z}) = \boldsymbol{y}$ is realized and observable; for $t \neq \boldsymbol{z}$ the outcome $\boldsymbol{y}(t)$ is counterfactual and unobservable. Hence, one can summarize the information embodied in this structure through a random set

$$\boldsymbol{Y}(t) = \begin{cases} \{\boldsymbol{y}\} & \text{if } t = \boldsymbol{z}, \\ \mathcal{Y} & \text{if } t \neq \boldsymbol{z}. \end{cases}$$

Measurability for all compact sets $K \subseteq \mathcal{Y}$ follows because

$$\{\boldsymbol{Y}(t) \cap K \neq \emptyset\} = \{\boldsymbol{y} \in K, \boldsymbol{z} = t\} \cup \{\mathcal{Y} \subseteq K, \boldsymbol{z} \neq t\} \in \mathfrak{A}.$$

One may observe that, when $\mathcal{Y} = [0, 1]$, this example is a special case of Example 1.11, with $\boldsymbol{y}_{\mathrm{L}} = \boldsymbol{y}\mathbf{1}_{\boldsymbol{z}=t}$ and $\boldsymbol{y}_{\mathrm{U}} = \boldsymbol{y}\mathbf{1}_{\boldsymbol{z}=t} + \mathbf{1}_{\boldsymbol{z}\neq t}$.

Example 1.14 (Binary endogenous variable in a binary model) Consider the model

$$\boldsymbol{y}_1 = \mathbf{1}_{\ell(\boldsymbol{y}_2) < \boldsymbol{u}},$$

where $\ell(\cdot)$ is an unknown function, $\boldsymbol{y}_1, \boldsymbol{y}_2$ are binary random variables taking values in $\{0, 1\}$, the marginal distribution of the random variable $\boldsymbol{u}$ is uniform on $[0, 1]$, but $\boldsymbol{u}$'s distribution conditional on $\boldsymbol{y}_2$ is otherwise unrestricted. The tuple $(\boldsymbol{y}_1, \boldsymbol{y}_2)$ is observed, while $\boldsymbol{u}$ is unobserved. Then

$$\boldsymbol{U} = U(\boldsymbol{y}_1, \boldsymbol{y}_2; \ell) = \begin{cases} [\ell(\boldsymbol{y}_2), 1] & \text{if } \boldsymbol{y}_1 = 1, \\ [0, \ell(\boldsymbol{y}_2)] & \text{if } \boldsymbol{y}_1 = 0 \end{cases}$$

is a random set that collects, for given $\boldsymbol{y}_1, \boldsymbol{y}_2$ and hypothesized function $\ell(\cdot)$, the values of $\boldsymbol{u}$ consistent with the model. In the definition of the function $U(\boldsymbol{y}_1, \boldsymbol{y}_2; \ell)$, we used that $\mathbf{P}\{\boldsymbol{u} = \ell(\boldsymbol{y}_2)\} = 0$ because $\boldsymbol{u}$ is uniformly distributed. For given compact set $K \subset \mathbb{R}$, we then have

$$\Big\{U(\boldsymbol{y}_1, \boldsymbol{y}_2; \ell) \cap K \neq \emptyset\Big\} = \Big\{[\ell(\boldsymbol{y}_2), 1] \cap K \neq \emptyset, \boldsymbol{y}_1 = 1\Big\} \cup \Big\{[0, \ell(\boldsymbol{y}_2)] \cap K \neq \emptyset, \boldsymbol{y}_1 = 0\Big\}.$$

Measurability follows from Example 1.11.

Example 1.15 (Entry game) Consider a two-player *entry game* where each player j can choose to enter the market ($y_j = 1$) or stay out of the market ($y_j = 0$). Let $\varepsilon_1, \varepsilon_2$ be two random variables, and let $\theta_1 \leq 0$ and $\theta_2 \leq 0$ be two parameters. Assume that the players' payoffs are given by

$$\pi_j = y_j(\theta_j y_{3-j} + \varepsilon_j), \quad j = 1, 2.$$

Each player enters the game if and only if $\pi_j \geq 0$. Then, for given values of θ_1 and θ_2, the set of pure strategy Nash equilibria, denoted $\boldsymbol{Y}_\theta$, is depicted in Figure 1.2 as a function of ε_1 and ε_2. The figure shows that, for $(\varepsilon_1, \varepsilon_2) \notin [0, -\theta_1) \times [0, -\theta_2)$, the equilibrium of the game is unique, while, for $(\varepsilon_1, \varepsilon_2) \in [0, -\theta_1) \times [0, -\theta_2)$, the game admits multiple equilibria and the corresponding realization of Y_θ has cardinality 2. An equilibrium is guaranteed to exist because we assume $\theta_1 \leq 0, \theta_2 \leq 0$. To see that $\boldsymbol{Y}_\theta$ is a random closed set, notice that, in this example, one can take $\mathfrak{X} = \{(0,0), (1,0), (0,1), (1,1)\}$, and that all its subsets are compact (see Example 1.7). Then

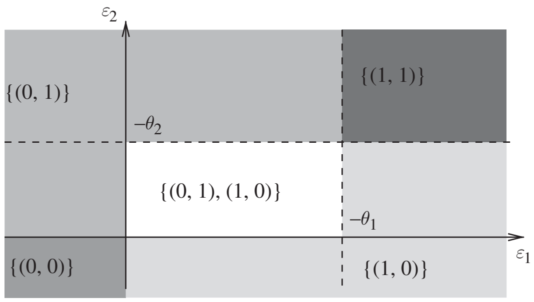

Figure 1.2 The set of pure strategy Nash equilibria of a two-player entry game as a function of ε_1 and ε_2.

$$\{\boldsymbol{Y}_\theta \cap K \neq \emptyset\} = \{(\varepsilon_1, \varepsilon_2) \in G_K\} \in \mathfrak{A},$$

where G_K is a set determined by the chosen K. This set gives the values of $(\varepsilon_1, \varepsilon_2)$ such that an equilibrium from K is feasible. For example, if $K = \{(0,0)\}$, then $G_K = (-\infty, 0) \times (-\infty, 0)$. If $K = \{(1,1)\}$, then $G_K = [\theta_1, \infty) \times [\theta_2, \infty)$. Measurability follows because ε_1 and ε_2 are random variables.

Example 1.16 (English auction) Consider an English auction where M bidders have continuously and independently distributed valuations $\tilde{\boldsymbol{v}}_1, \dots, \tilde{\boldsymbol{v}}_M$ of an item conditional on auction characteristics $\boldsymbol{z} = z$, with strictly increasing cumulative distribution function denoted $\mathsf{F}_{\tilde{\boldsymbol{v}}}(v|z)$. Suppose there is no minimum reserve price or minimum bid increment. Let $\boldsymbol{v} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_M)$ and $\boldsymbol{y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_M)$ denote respectively ordered valuations and ordered final bids so that almost surely $\boldsymbol{v}_1 \leq \boldsymbol{v}_2 \leq \cdots \leq \boldsymbol{v}_M$ and $\boldsymbol{y}_1 \leq \boldsymbol{y}_2 \leq \cdots \leq \boldsymbol{y}_M$. Note that $\boldsymbol{y}_i$ need not be the bid made by the bidder with valuation $\boldsymbol{v}_i$. In this example, the random vector $\boldsymbol{y}$ is observable but the random vector $\boldsymbol{v}$ is not. Let $\tilde{\boldsymbol{u}} \in [0,1]^M$ be M mutually independent uniform random variables with $\tilde{\boldsymbol{u}}$ independent with $\boldsymbol{z}$, and denote its order statistics $\boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_M)$. By the strict monotonicity of $\mathsf{F}_{\tilde{\boldsymbol{v}}}(v|z)$ in v, ordered valuations can be expressed as functions of these uniform order statistics as $\boldsymbol{v}_m = \mathsf{F}_{\tilde{\boldsymbol{v}}}^{-1}(\boldsymbol{u}_m|z)$, $m = 1, \dots, M$. Assume that (1) no one bids more than their valuations, and (2) no one allows an opponent to win at a price they are willing to beat. It can be shown that these two assumptions imply $\boldsymbol{v}_m \geq \boldsymbol{y}_m$ for all m and $\boldsymbol{y}_M \geq \boldsymbol{v}_{M-1}$. We can express these conditions as

$$\mathsf{F}_{\tilde{\boldsymbol{v}}}(\boldsymbol{y}_M|z) \geq \boldsymbol{u}_{M-1}, \quad \mathsf{F}_{\tilde{\boldsymbol{v}}}(\boldsymbol{y}_m|z) \leq \boldsymbol{u}_m \quad \forall\, m \in \{1, \dots, M\}.$$

The random vector $\boldsymbol{u}$ is supported on R_u, the orthoscheme of the unit M-cube in which $\boldsymbol{u}_1 \leq \boldsymbol{u}_2 \leq \cdots \leq \boldsymbol{u}_M$. Given $\boldsymbol{z} = z$, the collection of values for $(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_M)$ consistent with the observed vector of final bids and the model is

$$\begin{aligned}\boldsymbol{U} &= U(\boldsymbol{y}_1, \dots, \boldsymbol{y}_M; \mathsf{F}_{\tilde{\boldsymbol{v}}}(\cdot|z)) \\ &= \Big\{u \in R_u : \mathsf{F}_{\tilde{\boldsymbol{v}}}(\boldsymbol{y}_M|z) \geq u_{M-1},\ \mathsf{F}_{\tilde{\boldsymbol{v}}}(\boldsymbol{y}_m|z) \leq u_m \ \ \forall m \in \{1, \dots, M\}\Big\}.\end{aligned}$$

This is a random closed set, and measurability follows as a special case of Example 1.10.

Example 1.17 (Confidence set) Suppose a researcher observes an i.i.d. sample of random vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ and is interested in a parameter vector $\theta \in \Theta \subseteq \mathbb{R}^d$ that determines the relationship between the variables of interest. Consider a confidence set for θ defined as the level set of a non-negative sample criterion function $T_n(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n; \theta)$,

$$\mathrm{CS}_n = \{\theta \in \Theta : T_n(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n; \theta) \leq c_n\},$$

with c_n a properly chosen critical level. Suppose that $T_n(x_1, \dots, x_n; \cdot) : \Theta \mapsto \mathbb{R}_+$ is lower semicontinuous. It follows from Example 1.9 that CS_n is a random closed set.

Random Variables Associated with Random Sets

The imposed measurability definition implies that a number of important functionals of $\boldsymbol{X}$ are random variables.

Example 1.18 (Norm) As a first example of such functionals, define the norm of a non-empty set $\boldsymbol{X}$ in the Euclidean space $\mathbb{R}^d$ endowed with the Euclidean norm $\|\cdot\|$ as

$$\|\boldsymbol{X}\| = \sup\{\|x\| : \ x \in \boldsymbol{X}\}.$$

This definition allows for an infinite value of $\|\boldsymbol{X}\|$, which appears if $\boldsymbol{X}$ is not bounded. In order to show that $\|\boldsymbol{X}\|$ is a random variable (with values in the extended real line), note that

$$\{\|\boldsymbol{X}\| \le t\} = \{\boldsymbol{X} \subseteq B_t(0)\} \in \mathfrak{A}$$

for all $t \ge 0$, where $B_t(0)$ is the closed ball of radius t centered at the origin. In other words, $\|\boldsymbol{X}\|$ is the radius of the smallest centered (at the origin) ball that contains $\boldsymbol{X}$.

Example 1.19 (Support function) For given vector $u \in \mathbb{R}^d$ and random closed set $\boldsymbol{X}$, consider

$$h_{\boldsymbol{X}}(u) = \sup\{x^\top u : \ x \in \boldsymbol{X}\}, \quad u \in \mathbb{R}^d,$$

which is called the *support function* of $\boldsymbol{X}$. The support function may take infinite values: if $\boldsymbol{X}$ is empty, its support function is set to be $-\infty$. The argument u is often restricted to belong to the unit sphere in $\mathbb{R}^d$,

$$\mathbb{S}^{d-1} = \{u \in \mathbb{R}^d : \ \|u\| = 1\}.$$

To show that $h_{\boldsymbol{X}}(u)$ is a random variable (with values in the extended real line), note that, for all $t \ge 0$,

$$\{h_{\boldsymbol{X}}(u) \le t\} = \{\boldsymbol{X} \subseteq H_t(u)\} \in \mathfrak{A},$$

where $H_t(u)$ is the half-space defined as $H_t(u) = \{w : \ w^\top u \le t\}$.

Example 1.20 (Distance function) Another important random variable related to random closed sets in $\mathbb{R}^d$ is the *distance function* $\mathbf{d}(u, \boldsymbol{X})$ given by the infimum of the Euclidean distance between $u \in \mathbb{R}^d$ and points from $\boldsymbol{X}$. Then

$$\{\mathbf{d}(u, \boldsymbol{X}) \le t\} = \{\boldsymbol{X} \cap B_t(u) \ne \emptyset\}$$

is a measurable event for all $t \ge 0$, meaning that $\mathbf{d}(u, \boldsymbol{X})$ is a non-negative random variable.