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Part I

Conceptual foundations

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1

Newtonian cosmology

While General Relativity (GR) is the only theory of gravitation that can describe the Universe on large scales, Newtonian gravitation provides an illuminating approximation to relativistic cosmology. It is thus worthwhile to briefly consider classical, non-relativistic Newtonian gravitation and its application to cosmology, before diving into the complexity and abstraction of the general relativistic version.

1.1 Newtonian gravitation

Newton's theory of gravitation has been spectacularly successful in application to all sorts of problems over the last 300+ years. But it has conceptual shortcomings at its roots that render it inapplicable to all but the simplest approximations to reality when applied to strong gravitational fields, or on the scales encountered in cosmology.

First, the theory includes no mechanism for the transmittal of gravitational force from one mass to another. The theory was criticized for this 'action at a distance' requirement during Newton's lifetime, but he brushed off the criticisms with the observation that the theory worked even if we did not know exactly how. The large distances encountered in cosmological applications, however, require a theoretical underpinning that accommodates the transmittal of gravitational forces over billions of light years; and the apparent instantaneous nature of Newtonian gravitation over such distances seems suspect at best.

Second, Newtonian gravitation offers no explanation for the equality of gravitational and inertial mass. The gravitational force exerted by a mass M on a massive test object located a distance d away is

$$\vec{F} = -G\frac{Mm_{(\mathrm{G})}}{d^3}\vec{d}\ ,$$

where by $m_{(G)}$ is meant the gravitational mass of the test object, that appearing in Newton's law of gravitation (above). The dynamical response of the test object is the acceleration

$$\vec{a}_m = \frac{\vec{F}}{m_{(I)}} = -G\frac{M}{d^3}\frac{m_{(G)}}{m_{(I)}}\vec{d} ,$$

where $m_{(I)}$ is the inertial mass of the object, that appearing in Newton's second law of mechanics, $\vec{F} = m\vec{a}$. It is a well-observed fact that $m_{(G)} = m_{(I)}$ to a high degree of precision *for all massive objects,* so

$$\vec{a}_m = -G\frac{M}{d^3}\vec{d} ,$$

independent of the object's mass: a gravitational field accelerates all massive objects at the same rate, irrespective of their mass (or anything else).[1] Gravitation is the only fundamental force for which this is true, and Newton's theory offers no suggestion as to why this is so.

Third, Newtonian dynamics (including gravitation) is based on concepts of absolute space and time. Newton's second law ($F = m\vec{a}$), for instance, only works in inertial reference frames, those experiencing no acceleration. But acceleration relative to *what*? If you're the only thing in the Universe, how do you know if you're accelerating or not? The nineteenth century Austrian physicist Ernst Mach had an interesting answer to such questions, since embodied in Mach's Principle:[2] that the origin of inertia lay in the combined gravitation of all the Universe's contents, and that space of itself had no existence as a thing. This principle was an important one in Einstein's thinking leading to GR, but has since fallen out of favor with modern physicists who include fields as properties of space itself.

But for Newton the only answer was that there was an absolute space relative to which all accelerations could be measured. Similarly (if not quite so obviously), there must be an absolute time that applies to all of space. Einstein's Theory of Special Relativity (SR) showed that both of these absolute concepts were erroneous, thus largely undermining the fundamentals of Newtonian gravitation.

Fourth, Newton's theory of gravitation is manifestly incorrect in applications to strong gravitational fields. At the time Einstein took on the task of developing its successor the most worrisome and well-established discrepancy in Newtonian

[1] The subject of Galileo's (possibly apocryphal) experiment of dropping objects of different weights from the Leaning Tower of Pisa, to see if they hit the piazza below at the same time. A more compelling version of the experiment, without the complication of air resistance, was performed by Apollo astronauts on the Moon using a rock hammer and a feather as test objects.

[2] See, e.g., Sciama (1969), Graves (1971), Rindler (1977), Peebles (1993), and Ghosh (2000) for analysis of Mach's Principle and discussions of its relation to GR and its present standing in modern physics and cosmology.

gravitation had to do with the orbit of Mercury. That elliptical and inclined orbit, embedded deep within the Sun's gravitational field, precesses at a rate of $\Delta\theta = 5599''.74 \pm 0''.40$ per century, due mostly to the gravitational perturbations of other planets. But all attempts to model the precession from Newtonian gravitation fell short by about 43″ per century, a figure two orders of magnitude larger than the estimated observational uncertainty. By the beginning of the twentieth century this apparent failure of Newtonian gravitation had become sufficiently worrisome as to prompt several attempts to modify the theory itself. Einstein's demonstration that his General theory of Relativity successfully predicted this anomalous precession was an important factor in its reception by the scientific community. Observations during the intervening ~ 100 years have since revealed several other areas in which GR gives a correct answer where Newtonian gravitation does not (think: black holes!).

1.2 Universal expansion

Yet another problem with Newtonian gravitation arises in application to dynamical models of the Universe as a whole; i.e., cosmology. A gravitationally mediated expansion characterized by densities of gravitating matter and energy can be described by two sets of equations: one or more field equations relating mass/energy densities to gravitational potentials, and the resulting equation of motion for a test particle in that potential. In Newtonian mechanics these are, respectively,

Newtonian Gravitation

Field (Poisson's) Equation: $$\nabla^2 \Phi = 4\pi G\rho \,, \tag{1.1}$$

Equation of motion: $$\vec{\mathbf{a}} = -\vec{\nabla}\Phi \,, \tag{1.2}$$

where ρ is the mass density, Φ is the Newtonian gravitational potential, and $\vec{\mathbf{a}}$ is the acceleration of a test particle in the gravitational potential. But Newtonian gravitational potentials cannot be unambiguously defined in an infinite, homogeneous medium where, by symmetry, the potential must be the same everywhere. Since Φ has no gradient under such conditions, there can be no gravitational dynamics. And since current observations strongly support the world-view of an effectively infinite and homogeneous[3] Universe – as did conventional thinking

[3] Homogeneous on sufficiently large scales.

prior to the development of GR and modern astronomical observations – this constitutes a serious obstacle to applying Newtonian gravitation to cosmology.

The matter can be finessed by suitable adjustments to the Newtonian theory, usually in terms of changes to Poisson's Equation.[4] But for purposes of illustration and comparison with the relativistic model to be derived in Chapter 8, it suffices to confine the analysis to finite universes with centers and to employ a classical energy analysis. Thus: the total mechanical energy E of a mass m a distance d from a fixed, central mass M is

$$E = U + K = -G\frac{Mm}{d} + \frac{1}{2}m\dot{d}^2 \ ,$$
$$\Rightarrow \dot{d}^2 = 2G\frac{M}{d} + 2\left(E/m\right) \ .$$

We can cast this into a cosmological context by (1) re-introducing the universal expansion function $a(t)$ (Equation (I.1)) so that $d(t) = d_0\, a(t)$; and (2) replacing the central mass M with uniformly distributed mass density throughout the spherical volume encompassed by the two masses, so that $M = (4/3)\,\pi\,\rho d^3$. Using these relations and the mass conservation condition $\rho d^3 = \rho_0 d_0^3$ to eliminate M and d from the above energy equation yields a differential equation for the expansion function:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\frac{\rho_0}{a^3} + 2\frac{E/m}{a^2 d_0^2} \ . \tag{1.3}$$

Note that the left-hand side of this equation is the square of the Hubble parameter, $H \equiv \dot{a}/a$, as defined in the Introduction.

This is the equation of motion of an object falling upward in a static gravitational field. The form of its solution depends critically on the value of E. If $E > 0$ the structure is unbound: $\dot{a}^2 > 0$ at all times and the expansion may continue forever (at an ever-decreasing rate). This is an **open** expansion corresponding to velocities exceeding that of escape. But if $E < 0$ the structure is gravitationally bound: $\dot{a} \to 0$ at sufficiently large a and the expansion stops and reverses itself. This is a **closed** expansion corresponding to velocities less than that of escape. The **critical** case separating these two corresponds to $E = 0$ and represents an object with exactly the escape velocity. Graphical examples of $a(t)$ for all three cases are shown in Figure 1.1.

The fate of this Newtonian Universe can be discerned by comparing its mass density to its expansion rate. From Equation (1.3) at the current time (when $a = 1$),

$$E = 0 \quad \Rightarrow \quad H_0^2 = \frac{8\pi G}{3}\rho_0 \ ,$$

[4] See Section 9.2 of Rindler (1977) for examples.

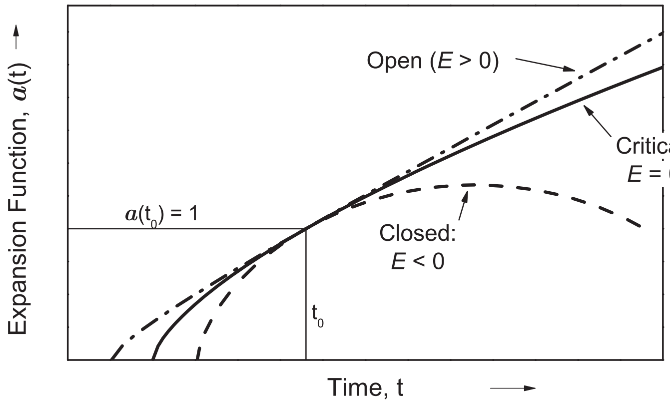

Figure 1.1 Sample expansion functions for Newtonian mechanics, all for the same value of H_0 (which is the slope of these functions at the current time, t_0).

where $H_0 = (\dot{a}/a)_{t_0} = \dot{a}(t_0)$ is the current value of the Hubble Parameter and $\rho_0 = \rho(t_0)$ is the current mass density. The critical mass density – that required for a zero-energy Universe currently expanding at the rate H_0 – is thus

$$\rho_{c,0} \equiv \frac{3H_0^2}{8\pi G} \,. \tag{1.4}$$

Inserting this into the expansion equation (1.3) for the current time and solving for the total energy:

$$2\frac{E/m}{d_0^2} = H_0^2\left(1 - \frac{\rho_0}{\rho_{c,0}}\right) \,. \tag{1.5}$$

The Newtonian Universe is open ($E > 0$) or closed ($E < 0$) depending upon whether its current mass density is less, or greater, respectively, than the critical density. Low density universes expand forever, high density ones eventually stop expanding and re-contract.

This is about as far as it is useful to carry the Newtonian analysis of cosmological expansion. While Equations (1.3) and (1.5) are gratifyingly similar to the fully relativistic expansion equations to be developed later in this book, and it is possible to join this Newtonian result to SR kinematics so as to produce a coherent picture of such things as cosmological redshifts; the logical sleights of hand employed in derivation of the Newtonian expansion equation largely invalidate its application to the real world. We need to invoke GR in order to realistically model the Universe on large scales, and that will require the considerable machinery developed in Parts II and III of this text.

Problems

1. Show that solutions to the expansion equation (1.3) in the critical case ($E = 0$) are of the form $a(t) \propto t^{2/3}$. Find an expression for the current time t_0 in such models in terms of the current mass density, ρ_0. Estimate the age of the Universe in this model in Gyr (10^9 years) if its mass density corresponds to 1 baryon (proton, neutron) per cubic meter, which is approximately what is currently observed.
2. Find an expression for the Hubble Parameter $H = \dot{a}/a$ as a function of time for the model of Problem 1. Find a numerical value for H for the current time and mass density of Problem 1, in units of both Gyr^{-1} and km/sec/Mpc.
3. Differentiate the Newtonian expansion equation (1.3) to derive an acceleration equation of the form $\ddot{a}/a$ as a function of the expansion function a, independent of the total energy. Show that this corresponds to a pure force of attraction.

2

General Relativity

Einstein's General Theory of Relativity (GR) was motivated principally by his desire to expand his very successful theory of Special Relativity (SR) to non-inertial reference frames. SR served to reconcile the invariance of the speed of light for all observers – as predicted by Maxwell's Equations of electromagnetism, and verified by the Michelson–Morley experiment – with Einstein's Principle of Special Relativity: that the laws of physics were the same in all non-accelerating reference frames. General Relativity, as Einstein envisioned it, would require the laws of physics to be identical in *all* reference frames, including accelerating ones. That this extension of the relativity principle leads to a theory of gravitation – which is what GR has become – was a consequence of the observed equality of gravitational and inertial mass: since *all* objects fell with the same acceleration in a given gravitational field, acceleration and gravitation are, in some sense, equivalent. Note that this singling out of gravitation distinguishes it from other fundamental forces, such as electromagnetism: acceleration and gravitation are connected in a unique manner.

But the details of that connection were totally non-obvious when Einstein set out to discover them; in particular, it did not seem possible at first to write laws of mechanics in a manner that is independent of the acceleration of the reference frame. In fact, Einstein never successfully united *all* forms of non-inertial motion into a single theory, but he did manage to do so with gravitation so that his General Relativity theory has effectively become one of gravitation, relegating Newton's theory of gravity to that of an approximation to the full relativistic theory. In particular, it is Einstein's theory of gravity that must be employed on the scales encountered in cosmology for a successful theory of the Universe's large-scale structure and evolution to be constructed.

The fundamental concepts underlying Einstein's theory of gravitation are these three: **General Covariance**, which expresses the relativity principle, that the laws of physics take the same form in all reference frames; **Equivalence**, which embodies the equality of gravitational and inertial mass; and **Space-Time**

Curvature, which provides the means by which gravitation controls dynamics. These are conceptually summarized in this chapter and are each the detailed subject of a separate chapter in Part II of this text.

2.1 Covariance

The Principle of General Covariance, as Einstein expressed it, is that the laws of physics are independent of our choices of reference frames or of coordinate systems, and that the equations of physics, properly constructed, should take the same form in all coordinate systems. Insistence on this property in development of SR proved to be crucial in extending SR to all of physics, including electromagnetism, in inertial reference frames. What worked so successfully in SR would apparently be a good choice for development of GR, at least in the basic stages.

The most obvious way to free physical laws from specific coordinate systems is to make all expressions of physical quantities overtly independent of coordinates. Thus, the Newtonian expression for gravitational potential Φ can be concisely written as Poisson's Equation: $\nabla^2\Phi = 4\pi G\rho$, which is true in all coordinate systems. But such simple representations are not readily extendable to mechanics in general, so instead we employ *co*variant forms which, while changing with coordinate systems, do so all in the same manner so that mathematical expressions of equalities of physical quantities remain unchanged even when coordinate systems change. Thus: if we have, say, a generally covariant vector equation of the form $A_i = B_i$ in one coordinate system and we change to another system, so that $A_i \to A'_i$ and $B_i \to B'_i$, it will nonetheless remain that $A'_i = B'_i$ as in the original coordinate system. We say that generally covariant vector components are *not in*variant – they *do* change with changes in coordinate systems – but they *are co*variant – they all change in the same manner so as to preserve their equality.

Equations written entirely in terms of generally covariant quantities remain true in all coordinate systems, including accelerating ones.

That general covariance is a special quality can be seen by considering the equations of motion of a force-free particle: $d^2x^i/dt^2 = 0$ for all coordinates x^i. This equation is manifestly untrue in, say, a rotating coordinate system defined by $\bar{x}^i = x^i \cos(\omega t)$, so that the simple equations of Newtonian mechanics are not generally covariant. To make them so we must write Newton's laws as, e.g., $\vec{\mathsf{a}} = 0$ in which $\vec{\mathsf{a}}$ assumes different (implied) forms in different coordinate systems: in accelerating systems this would include such complications as Coriolis and centripedal accelerations.

It turns out that all equations of physics may be written in a generally covariant form if one is willing to accept very complicated expressions, but as a practical