# Part I

## Fractional Sobolev spaces

# 1

# Fractional framework

Recently, great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for pure mathematical research and in view of concrete real-world applications. This type of operator arises in a quite natural way in many different contexts, such as, among others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, chemical reactions of liquids, population dynamics, geophysical fluid dynamics, and mathematical finance (American options). The fractional Laplacian also provides a simple model to describe certain jump Lévy processes in probability theory. In all these cases, the nonlocal effect is modeled by the singularity at infinity. For more details and applications, see [13, 35, 47, 52, 55, 75, 140, 216, 217, 218, 219] and the references therein.

From a physical point of view, nonlocal operators play a crucial rule in describing several phenomena. As a general reference in this topic, we cite the recent paper of Vázquez [217]. In that paper, the author describes two models of flow in porous media, including nonlocal (long-range) diffusion effects, providing a long list of references related to physical phenomena and nonlocal operators. The first model is based on Darcy's law, and the pressure is related to the density by an inverse fractional Laplacian operator. The second model is more in the spirit of fractional Laplacian flows but nonlinear: contrary to the usual porous medium flows, it has infinite speed of propagation.

Moreover, the fractional power of the Laplace operator has been studied in relation to the obstacle problem that appears in many contexts, such as in the study of anomalous diffusion, in the so-called quasi-geostrophic flow problem, and in pricing of American options governed by assets evolving according to jump processes (see, e.g., the papers [54, 188, 189]).

For the sake of completeness, we mention that fractional nonlocal problems have been considered recently under certain Neumann boundary conditions using different methods and approaches (see, among others, the papers [78, 79, 80, 160, 209]). All these different Neumann problems for nonlocal operators recover the classical Neumann problem as a limit case, and most of them have clear probabilistic interpretations as well. In this setting, in [85], the authors propose an intriguing approach to studying Neumann problems with a variational structure.

In this chapter we sketch the basic facts on fractional Sobolev spaces and fractional nonlocal operators. Our treatment is mostly self-contained, and we tacitly assume that the reader has some knowledge of the basic objects discussed here. More precisely, the main purpose of this section is to present some results on fractional Sobolev spaces and nonlocal operators in the form in which they will be exploited later on. Since this is an introductory chapter to convey the framework we work in, the rigorous proofs will be kept to a minimum. Some extra reading of the references may be necessary to truly learn the material.

Here we will consider a nonlocal fractional framework, providing models and theorems related to nonlocal phenomena. The results of this chapter are based on the papers [83, 147, 148, 198, 199, 200].

## 1.1 Fourier transform of tempered distributions

In this section we just recall briefly the notion of Fourier transform of a tempered distribution. First of all, we consider the Schwartz space $\mathscr{S}$ of rapidly decaying $C^\infty(\mathbb{R}^n)$ functions whose topology is generated by the seminorms $\{p_j\}_{j\in\mathbb{N}}$ defined as

$$p_j(\varphi) := \sup_{x\in\mathbb{R}^n} (1+|x|)^j \sum_{|\alpha|\leq j} |D^\alpha \varphi(x)|,$$

where $\varphi \in \mathscr{S}(\mathbb{R}^n)$. More precisely, $\mathscr{S}$ contains the smooth functions $\varphi$ satisfying

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < +\infty,$$

for all multi-indices $\alpha$ and $\beta \in \mathbb{N}_0^n$.

The natural locally convex topology on $\mathscr{S}$ can be characterized by the following notion of convergence:

*the sequence* $\{\varphi_j\}_{j\in\mathbb{N}}$ *converges to* 0 *in* $\mathscr{S}$ *if and only if*

$$\lim_{j\to+\infty} x^\alpha D^\beta \varphi_j(x) = 0, \ \textit{for all } \alpha \textit{ and } \beta \in \mathbb{N}_0^n.$$

We denote by

$$\mathscr{F}\varphi(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \varphi(x)\,dx \tag{1.1}$$

the *Fourier transform* of a function $\varphi \in \mathscr{S}$. Note that, for every $\varphi \in \mathscr{S}$, one has that $\mathscr{F}\varphi \in \mathscr{S}$.

It may be readily verified that the Fourier transform (1.1) and the inverse Fourier transform, given by

$$\mathscr{F}^{-1}\varphi(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi\cdot}\varphi(\xi)\,d\xi, \tag{1.2}$$

are both continuous on $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$. Moreover, since

$$\mathscr{F}^{-1}\mathscr{F}\varphi = \mathscr{F}\mathscr{F}^{-1}\varphi = \varphi,$$

each of them is, in fact, an isomorphism and a homeomorphism of $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$.

Now let $\mathscr{S}'$ be the topological dual of $\mathscr{S}$. As usual, a tempered distribution is an element of $\mathscr{S}'$. If $T \in \mathscr{S}'$, the Fourier transform of $T$ can be defined as the tempered distribution given by

$$\langle \mathscr{F}T, \varphi\rangle := \langle T, \mathscr{F}\varphi\rangle,$$

for every $\varphi \in \mathscr{S}$, where $\langle\cdot,\cdot\rangle$ denotes the usual duality bracket between $\mathscr{S}$ and its dual $\mathscr{S}'$.

By using definition (1.1), one has

$$u \in L^2(\mathbb{R}^n) \quad \text{if and only if} \quad \mathscr{F}u \in L^2(\mathbb{R}^n) \tag{1.3}$$

and

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\mathscr{F}u\|_{L^2(\mathbb{R}^n)}, \tag{1.4}$$

for every $u \in L^2(\mathbb{R}^n)$. Formula (1.4) is the so-called Parseval–Plancherel formula, which will be crucial in what follows for proving the equivalence between the fractional spaces $H^s(\mathbb{R}^n)$ and $\widehat{H}^s(\mathbb{R}^n)$ (see Corollary 1.15).

For a detailed introduction to the classical theory of distribution and Fourier transform, we refer to the monograph [187] and the recent book [69] for several applications to elliptic problems of linear and nonlinear functional analysis.

## 1.2 Fractional Sobolev spaces

Let $\Omega$ be a possibly nonsmooth, open set of the Euclidean space $\mathbb{R}^n$ and $p \in [1, +\infty)$. For any $s > 0$, we would define the fractional Sobolev space $W^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also called *Aronszajn*, *Gagliardo*, or *Slobodeckij spaces*, by the names of the ones who introduced them, almost simultaneously (see [15, 110, 207]).

If $s \geq 1$ is a positive integer, we denote by $W^{s,p}(\Omega)$ the classical Sobolev space equipped with the standard norm

$$\|u\|_{W^{s,p}(\Omega)} := \sum_{0\leq|\alpha|\leq s} \|D^\alpha u\|_{L^p(\Omega)},$$

for every $u \in W^{s,p}(\Omega)$, where here and in what follows $\|\cdot\|_{L^p(\Omega)}$ denotes the usual norm in $L^p(\Omega)$, and $D^\alpha$ stands for the $\alpha$-distributional derivative. This section is devoted to the definition of fractional Sobolev spaces; that is, here we are interested in the case where $s \notin \mathbb{N}$.

For a fixed $s \in (0,1)$, we recall that the Sobolev space $W^{s,p}(\Omega)$ is defined as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{n/p+s}} \in L^p(\Omega \times \Omega) \right\}.$$

It is endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left( \int_\Omega |u(x)|^p \, dx + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p}, \tag{1.5}$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left( \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p} \tag{1.6}$$

is the *Gagliardo seminorm* of $u$.

When $s > 1$ and $s \notin \mathbb{N}$, we can write $s = m + \sigma$, where $m \in \mathbb{N}$ and $\sigma \in (0,1)$. We can define $W^{s,p}(\Omega)$ as follows:

$$W^{s,p}(\Omega) := \left\{ u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha \text{ s.t. } |\alpha| = m \right\}.$$

In this case, $W^{s,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left( \|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{1/p},$$

for every $u \in W^{s,p}(\Omega)$. All in all, the space $W^{s,p}(\Omega)$ is well defined and is a Banach space for every $s > 0$.

As in the classical case (i.e., $s \in \mathbb{N}$), any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support. Indeed, for any $s > 0$,

$$\overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^n)}} = W^{s,p}(\mathbb{R}^n);$$

that is, the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.

In general, if $\Omega \subset \mathbb{R}^n$, the space $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. Hence, we denote by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$; that is,

$$W_0^{s,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}}.$$

With this definition, we can also construct $W^{s,p}(\Omega)$ when $s < 0$. Indeed, for $s < 0$ and $p \in (1,+\infty)$, we can define

$$W^{s,p}(\Omega) := \left(W_0^{-s,q}(\Omega)\right)';$$

that is, $W^{s,p}(\Omega)$ is the dual space of $W_0^{-s,q}(\Omega)$, where $1/p + 1/q = 1$.

### 1.2.1 Embedding properties

This subsection is devoted to the embeddings of fractional Sobolev spaces into Lebesgue spaces. We point out that Sobolev inequalities and continuous (compact) embeddings of the spaces $W^{s,p}$ into the classical Lebesgue spaces $L^q$ are exhaustively treated in [83, sections 6 and 7] (see also [3]). Here we recall briefly some basic facts.

**Proposition 1.1** Let $p \in [1,+\infty)$ and let $\Omega$ be an open set in $\mathbb{R}^n$. Then the following assertions hold true:

(a) If $0 < s \le s' < 1$, then the embedding
$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$
is continuous. Hence, there exists a constant $C_1(n,s,p) \ge 1$ such that
$$\|u\|_{W^{s,p}(\Omega)} \le C_1(n,s,p)\|u\|_{W^{s',p}(\Omega)},$$
for any $u \in W^{s',p}(\Omega)$.

(b) If $0 < s < 1$ and $\Omega$ is of class $C^{0,1}$ (i.e., with Lipschitz boundary) and with bounded boundary $\partial\Omega$, then the embedding
$$W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$
is continuous. Hence, there exists a constant $C_2(n,s,p) \ge 1$ such that
$$\|u\|_{W^{s,p}(\Omega)} \le C_2(n,s,p)\|u\|_{W^{1,p}(\Omega)},$$
for any $u \in W^{1,p}(\Omega)$.

(c) If $s' \ge s > 1$ and $\Omega$ is of class $C^{0,1}$, then the embedding
$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$
is continuous.

*Proof* See propositions 2.1 and 2.2 and corollary 2.3 in [83]. □

Now we recall some basic properties about continuous (compact) embeddings of the fractional Sobolev spaces into Lebesgue spaces. In what follows, we need the following definition:

**Definition 1.2** For any $s \in (0,1)$ and any $p \in [1,+\infty)$, an open set $\Omega \subset \mathbb{R}^n$ is an *extension domain* for $W^{s,p}$ if there exists a positive constant $C := C(n,p,s,\Omega)$ such that *for every function $u \in W^{s,p}(\Omega)$, there exists $\mathscr{E}_u \in W^{s,p}(\mathbb{R}^n)$ such that $\mathscr{E}_u(x) = u(x)$ for any $x \in \Omega$ and*
$$\|\mathscr{E}_u\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)}.$$

Note that any open set of class $C^{0,1}$ with bounded boundary is an extension domain for $W^{s,p}(\mathbb{R}^n)$; see [83, theorem 5.4] for a direct proof.

For the sake of completeness, we also recall an interesting result, proved in [83, lemma 5.1], about the construction of the extension $\mathscr{E}_u$ to the whole of $\mathbb{R}^n$ of a function $u$ defined on an open set $\Omega \subset \mathbb{R}^n$.

**Lemma 1.3** *Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $u \in W^{s,p}(\Omega)$ with $s \in (0,1)$ and $p \in [1,+\infty)$. If there exists a compact subset $\mathscr{K} \subset \Omega$ such that $u \equiv 0$ in $\Omega \setminus \mathscr{K}$, then the extension function $\mathscr{E}_u$, defined as*

$$\mathscr{E}_u(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

*belongs to $W^{s,p}(\mathbb{R}^n)$, and*

$$\|\mathscr{E}_u\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)},$$

*where $C$ is a suitable positive constant depending on $n, p, s, \mathscr{K}$, and $\Omega$.*

Now we are ready to discuss the embedding properties of $W^{s,p}$. For this purpose, we distinguish three different cases, that is, $sp < n$, $sp = n$, and $sp > n$. We refer to [83, sections 6–8] for a proof of these results.

**Case 1:** $sp < n$.

**Theorem 1.4** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < n$. Then there exists a positive constant $C := C(n,p,s)$ such that, for any $u \in W^{s,p}(\mathbb{R}^n)$,*

$$\|u\|^p_{L^{p_s^*}(\mathbb{R}^n)} \le C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy,$$

*where the constant*

$$p_s^* := \frac{pn}{n-sp}$$

*is the so-called fractional critical exponent. Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p_s^*]$. Moreover, the embedding $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is compact for every $q \in [p, p_s^*)$.*

In an extension domain $\Omega$, the following embedding result holds:

**Theorem 1.5** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < n$. Let $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C := C(n,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{s,p}(\Omega)},$$

*for any $q \in [p, p_s^*]$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p_s^*]$. If, in addition, $\Omega$ is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*)$.*

**Case 2:** $sp = n$**.**

**Theorem 1.6** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = n$. Then there exists a positive constant $C := C(n,p,s)$ such that, for any $u \in W^{s,p}(\mathbb{R}^n)$,*

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\mathbb{R}^n)},$$

*for any $q \in [p,+\infty)$; that is, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p,+\infty)$.*

For an extension domain $\Omega$, we have the following embedding result:

**Theorem 1.7** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = n$. Let $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C := C(n,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

*for any $q \in [p,+\infty)$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p,+\infty)$. If, in addition, $\Omega$ is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1,+\infty)$.*

**Case 3:** $sp > n$**.** Here $C^{0,\alpha}(\Omega)$ denotes the space of Hölder continuous functions, endowed with the standard norm

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

**Theorem 1.8** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp > n$. Let $\Omega$ be a $C^{0,1}$ domain of $\mathbb{R}^n$. Then there exists a positive constant $C := C(n,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

*with $\alpha := (sp-n)/p$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $C^{0,\alpha}(\Omega)$.*

The preceding regularity property remains valid for functions in $W^{s,p}$ when $sp > n$ and $\Omega$ is an extension domain for $W^{s,p}$ with no external cusps (see [83, theorem 8.2]). As a consequence of Theorem 1.8, we have the following result:

**Corollary 1.9** Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp > n$. Let $\Omega$ be a $C^{0,1}$ bounded domain of $\mathbb{R}^n$. Then the embedding

$$W^{s,p}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$$

is compact for every $\beta < \alpha$, with $\alpha := (sp-n)/p$.

*Proof* Let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $W^{s,p}(\Omega)$. Theorem 1.8 ensures that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $C^{0,\alpha}(\Omega)$. Hence, there exists $C > 0$ such that

$$\|u_j\|_{L^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u_j(x) - u_j(y)|}{|x-y|^\alpha} \leq C \quad \forall\, j \in \mathbb{N}. \tag{1.7}$$

By using (1.7), the Ascoli–Arzelà theorem gives that

$$u_j \to u_\infty \quad \text{uniformly in } \Omega \tag{1.8}$$

as $j \to +\infty$, for some $u_\infty \in C(\Omega)$. Moreover, (1.7) and (1.8) give

$$|u_\infty(x) - u_\infty(y)| = \lim_{j\to+\infty} |u_j(x) - u_j(y)| \leq C|x-y|^\alpha, \tag{1.9}$$

for any $x, y \in \Omega$. Thus, the function $u_\infty$ belongs to $C^{0,\alpha}(\Omega)$.

Let us prove that $u_j \to u_\infty$ in $C^{0,\beta}(\Omega)$ as $j \to +\infty$, for every $\beta < \alpha$. Taking into account (1.8), we have to show that

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|(u_j - u_\infty)(x) - (u_j - u_\infty)(y)|}{|x-y|^\beta} \to 0 \tag{1.10}$$

as $j \to +\infty$. Since

$$\|u_j - u_\infty\|_{L^\infty(\Omega)} \to 0 \quad \text{as } j \to +\infty,$$

for every $\varepsilon > 0$, there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\|u_j - u_\infty\|_{L^\infty(\Omega)} \leq \frac{\varepsilon}{2}\left(\frac{\varepsilon}{2C}\right)^{\beta/(\alpha-\beta)} \qquad \forall\, j \geq j_\varepsilon. \tag{1.11}$$

Now inequalities (1.7) and (1.9) give

$$|(u_j - u_\infty)(x) - (u_j - u_\infty)(y)| \leq 2C|x-y|^{\alpha-\beta}|x-y|^\beta \quad \forall\, x, y \in \Omega. \tag{1.12}$$

If $2C|x-y|^{\alpha-\beta} < \varepsilon$, inequality (1.12) ensures that

$$|(u_j - u_\infty)(x) - (u_j - u_\infty)(y)| \leq \varepsilon|x-y|^\beta \quad \forall\, x, y \in \Omega. \tag{1.13}$$

Here we use the fact that $\beta < \alpha$.