PART 3

Lagrangian intersection Floer homology

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Floer homology on cotangent bundles

In the 1960s, Arnol'd first predicted (Ar65) the existence of *Lagrangian intersection theory* (on the cotangent bundle) as the intersection-theoretic version of the Morse theory and posed *Arnol'd's conjecture*: the geometric intersection number of the zero section of T^*N for a compact manifold N is bounded from below by the one given by the number of critical points provided by the Morse theory on N. This original version of the conjecture is still open due to the lack of understanding of the latter Morse-theoretic invariants. However, its cohomological version was proven by Hofer (H85) using the *direct approach* of the classical variational theory of the action functional. This was inspired by Conley and Zehnder's earlier proof (CZ83) of Arnol'd's conjecture on the number of fixed points of Hamiltonian diffeomorphisms. Around the same time Chaperon (Ch84) and Laudenbach and Sikorav (LS85) used the *broken geodesic* approximation of the action functional and the method of *generating functions* in their proof of the same result. This replaced Hofer's complicated technical analytic details by simple more or less standard Morse theory.

The proof published by Chaperon and by Laudenbach and Sikorav is reminiscent of Conley and Zehnder's proof (CZ83) in that both proofs reduce the infinite-dimensional problem to a finite-dimensional one. (Laudenbach and Sikorav's method of generating functions was further developed by Sikorav (Sik87) and then culminated in Viterbo's theory of *generating functions quadratic at infinity* (Vi92).)

In the meantime, Floer introduced in (Fl88b) a general infinite-dimensional homology theory, now called the Floer homology, which is based on the study of the moduli space of an elliptic equation of the Cauchy–Riemann type that occurs as the L^2-gradient flow of the action integral associated with the variational problem. In particular Hofer's theorem mentioned above is a special case of Floer's (Fl88a) (at least up to the orientation problem, which was solved later in (Oh97b)), if we set $L_0 = \phi(o_N)$, $L_1 = o_N$ in the cotangent bundle. (Floer's construction is applicable not only to the action functional in symplectic geometry but also to the various first-order elliptic systems that appear in low-dimensional topology, e.g., the anti-self-dual Yang–Mills equation and the Seiberg–Witten monopole equation, and has been a fundamental ingredient in recent developments in low-dimensional topology as well as in symplectic topology.)

In (Oh97b), the present author exploits the natural filtration present in the Floer complex associated with the classical action functional and provides a Floer-theoretic construction of Viterbo's invariants. This construction is partially motivated by Weinstein's observation that the classical action functional

$$\mathcal{A}_H(\gamma) = \int_\gamma p\, dq - \int_0^1 H(t, \gamma(t))dt$$

is a canonical 'generating function' of the Lagrangian submanifold $\phi_H^1(L)$.

In this chapter we give a brief summary of the Floer theory on cotangent bundles in which we can illustrate essentially all the aspects of known applications of Floer homology to symplectic topology, which does not require a study of the bubbling phenomenon because bubbling does not occur. We postpone the study of this crucial aspect of the bubbling phenomenon and its technical underpinning in general to later chapters.

12.1 The action functional as a generating function

We start with the first variation formula (3.4.15) restricted to the case of cotangent bundles with $\alpha = -\theta$, $\theta = \sum_{i=1}^n p_i\, dq^i$ in which the Liouville one-form

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 \omega_0(\dot\gamma(t) - X_H(t, \gamma(t)), \xi(t))dt + \langle\theta(\gamma(1)), \xi(1)\rangle \\ - \langle\theta(\gamma(0)), \xi(0)\rangle \tag{12.1.1}$$

for $\gamma : [0, 1] \to T^*N$ and $\xi \in \Gamma(\gamma^*T(T^*N))$ is a vector field along γ.

A moment's reflection on this formula gives rise to several important consequences.

First, we consider the set of paths $\gamma : [0, 1] \to T^*N$ issued at a point in the zero section. We denote

$$\Omega(0; o_N) = \{\gamma \mid \gamma(0) \in o_N\}.$$

There is a natural map $\pi\mathrm{ev}_1 : \Omega(0; o_N) \to N$ defined by

$$\pi \circ \mathrm{ev}_1(\gamma) = \pi(\gamma(1)). \tag{12.1.2}$$

This defines a fibration of $\Omega(0; o_N)$ over N whose fiber at $q \in N$ is given by

$$\Omega(o_N, T_q^*N) = \{\gamma \mid \gamma(0) \in o_N,\ \gamma(1) \in T_q^*N\}.$$

The fiber derivative of $\mathcal{A}_H$ for $\pi\mathrm{ev}_1$ at q is nothing but the first variation of $\mathcal{A}_H : \Omega(o_N, T_q^*N) \to \mathbb{R}$. This shows that we have

$$d^{\mathrm{fiber}}\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \omega_0(\dot\gamma(t) - X_H(t, \gamma(t)), \xi(t))\,dt$$

for all $\xi \in T_\gamma\Omega(o_N, T_q^*N)$. Therefore the fiber critical set thereof, denoted by $\Sigma_{\mathcal{A}_H} \subset \Omega(0; o_N)$, is given by the set of solutions of Hamilton's equation

$$\dot x = X_H(t, x),\ x(0) \in o_N,\ x(1) \in T_q^*N.$$

We note that on $\Sigma_{\mathcal{A}_H}$ we have

$$d\mathcal{A}_H(\gamma)(\xi) = \langle \gamma(1), d\pi(\xi(1))\rangle$$

from (12.1.1).

Exercise 12.1.1 Complete a formal heuristic argument to derive from this formula that the push-forward of the one-form $d\mathcal{A}_H(\gamma)$ is nothing but $\gamma(1) \in T_q^*N$.

This completes the heuristic proof of the following proposition, which was observed by Weinstein.

Proposition 12.1.2 (Weinstein) *The pair $(\mathcal{A}_H, \Omega(0; o_N))$ is a generating function of $\phi_H^1(o_N)$ in the above sense.*

In fact, there is the natural finite-dimensional reduction of this generating function which we call the *basic generating function* of $L_H = \phi_H^1(o_N)$ and denote by $h_H : L_H \to \mathbb{R}$. For given $x \in L_H$, we denote

$$z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x)),$$

which is a Hamiltonian trajectory such that

$$z_x^H(0) \in o_N, \quad z_x^H(1) = x \tag{12.1.3}$$

by definition. We recall the following basic lemmata used in (Oh97b), whose proofs we leave as an exercise.

Lemma 12.1.3 *The function $h_H : L_H \to \mathbb{R}$ defined by*

$$h_H(x) = \mathcal{A}_H(z_x^H), \quad z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$$

satisfies

$$i_H^* \theta = dh_H,$$

i.e., h_H is a generating function of L_H. We call h_H the basic generating function *of L_H.*

Exercise 12.1.4 Prove this lemma.

Another consequence of the formula (12.1.1) is the characterization of the natural boundary conditions for the variational theory of $\mathcal{A}_H$. One obvious natural boundary condition is the periodic boundary condition $\gamma(0) = \gamma(1)$. A more interesting class of natural boundary conditions is the following.

Proposition 12.1.5 *Consider the conormal bundle $i_S : \nu^* S \hookrightarrow T^* N$ for a submanifold $S \subset N$. Then $i_S^* \theta = 0$.*

Proof Let $\xi \in T_\alpha(T^* N)$ and $\pi(\alpha) = x$. We have

$$i_S^* \theta(\xi) = \alpha(x)(d\pi(\xi(\alpha)).$$

But this pairing vanishes because $d\pi(\xi(\alpha)) \in T_x S$ and $\alpha(x) \in \nu_x^* S$ by definition. This finishes the proof. □

By this proposition, if we restrict the action functional $\mathcal{A}_H$ to the subset

$$\Omega_{S_0 S_1} = \Omega(\nu^* S_0, \nu^* S_1) = \{\gamma : [0, 1] \to T^* N \mid \gamma(0) \in \nu^* S_0,\ \gamma(1) \in \nu^* S_1\}$$

the first variation of $d\mathcal{A}_H$ is reduced to

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \omega_0(\dot\gamma(t) - X_H(t, \gamma(t)), \xi(t))dt. \tag{12.1.4}$$

We point out that the path space $\Omega(o_N, T_q^* N)$ defined above is a special case of a conormal boundary condition corresponding to $S_0 = N$ and $S_1 = \{q\}$.

An immediate corollary of Proposition 12.1.5 is the following characterization of the critical-point equation: $d\mathcal{A}_H|_{\Omega(S_0, S_1)}(\gamma) = 0$ if and only if γ satisfies

$$\dot\gamma = X_H(t, \gamma(t)), \quad \gamma(0) \in \nu^* S_0,\ \gamma(1) \in \nu^* S_1. \tag{12.1.5}$$

Definition 12.1.6 (Action spectrum) We define $\mathrm{Spec}(H; S_0, S_1)$ to be the set of critical values and call it the action spectrum of $\mathcal{A}_H$ on $\Omega_{S_0 S_1}$, i.e.,

$$\mathrm{Spec}(H; S_0, S_1) = \{\mathcal{A}_H(z) \mid \dot z = X_H(t, z(t)), \quad z(0) \in \nu^* S_0,\ z(1) \in \nu^* S_1\}.$$

Proposition 12.1.7 *The subset* $\mathrm{Spec}(H; S_0, S_1) \subset \mathbb{R}$ *is compact and has measure zero.*

Proof We have a one-to-one correspondence between the solutions of (12.1.5) and the intersection

$$\phi_{H^1}(\nu^* S_0) \cap \nu^* S_1,$$

as shown in Section 3.1. Clearly this set is compact. On the other hand, the function from $\nu^* S_1$ to $\mathbb{R}$

$$h : x \mapsto z_x^H \mapsto \mathcal{A}_H(z_x^H)$$

is smooth and hence its image of $\phi_{H^1}(\nu^* S_0) \cap \nu^* S_1$ is compact.

Exercise 12.1.8 The set of critical points of h coincides with the intersection set $\phi_{H^1}(\nu^* S_0) \cap \nu^* S_1$ and the set of critical values of h with $\mathrm{Spec}(H; S_0, S_1)$.

Sard's theorem applied to the smooth function $h : \nu^* S_1 \to \mathbb{R}$ then finishes the proof. □

We also have the transversality result that $\phi_{H^1}(\nu^* S_0) \pitchfork \nu^* S_1$ if and only if the linearization operator of (12.1.5)

$$\nabla_{\gamma'} - DX_H(\gamma) : \Gamma_{S_0 S_1}(\gamma^*(T(T^*N))) \to \Gamma(\gamma^* T(T^*N)) \tag{12.1.6}$$

is surjective, where $\Gamma_{S_0 S_1}(\gamma^*(T(T^*N)))$ is the subset of $\Gamma(\gamma^* T(T^*N))$ defined by

$$\Gamma_{S_0 S_1}(\gamma^*(T(T^*N))) = \{\xi \in \Gamma(\gamma^* T(T^*N)) \mid \\ \xi(0) \in T_{\gamma(0)}(\nu^* S_0),\ \xi(1) \in T_{\gamma(1)}(\nu^* S_1)\}.$$

The subset $\Gamma_{S_0 S_1}(\gamma^*(T(T^*N)))$ is also the tangent space $T_\gamma \Omega_{S_0 S_1}$ of $\Omega_{S_0 S_1}$ at γ. This correspondence is the key to the relationship between the *dynamics* of Hamilton's equations and the *geometry* of Lagrangian intersections.

12.2 L^2-gradient flow of the action functional

We first note that, if a Riemannian metric g is given to N, the associated Levi-Civita connection induces a natural almost-complex structure on T^*N, which we denote by J_g and call the *canonical almost-complex structure* (in terms of the metric g on N).

Definition 12.2.1 Let g be a Riemannian metric on N. The canonical almost complex structure J_g on T^*N is defined as follows. Consider the splitting

$$T_{(q,p)}(T^*N) = T^h_{(q,p)}(T^*N) \oplus T^v_{(q,p)}(T^*N)$$

with respect to the Levi-Civita connection of g. For every $(q,p) \in T^*N$, J_g maps the horizontal unit tangent vectors to vertical unit vectors.

We will fix the Riemannian metric g on N once and for all. We leave the proof of the following proposition as an exercise.

Proposition 12.2.2 *We have the following.*

(1) *J_g is compatible with the canonical symplectic structure ω_0 of T^*N.*
(2) *On the zero section $o_N \subset T^*N \cong T_{(q,0)}o_N$, J_g assigns to each $v \in T_qN \subset T_{(q,0)}(T^*N)$ the cotangent vector $J_g(v) = g(v,\cdot) \in T^*_qN \subset T_{(q,0)}(T^*N)$. Here we use the canonical splitting*

$$T_{(q,0)}(T^*N) \cong T_qN \oplus T^*_qN.$$

(3) *The metric $g_{J_g} := \omega_0(\cdot, J_g\cdot)$ on T^*N defines a Riemannian metric that has bounded curvature and injectivity radius bounded away from 0.*
(4) *J_g is invariant under the anti-symplectic reflection $\mathfrak{r} : T^*N \to T^*N$ mapping $(q,p) \mapsto (q,-p)$.*

We consider the class of compatible almost-complex structures J on T^*N such that

$$J \equiv J_g \text{ outside a compact set in } T^*N,$$

and denote the class by

$$\mathcal{J}^c := \{J \mid J \text{ is compatible to } \omega \text{ and } J \equiv J_g \\ \text{outside a compact subset in } T^*N\}.$$

We define and denote the *support* of J by

$$\operatorname{supp} J := \text{ the closure of } \{x \in T^*N \mid J(x) \neq J_g(x)\}.$$

Define

$$\mathcal{P}(\mathcal{J}^c) := C^\infty([0,1], \mathcal{J}^c) = \{J : [0,1] \to \mathcal{J}^c \mid J = \{J_t\}_{0 \le t \le 1}\}.$$

For each given $J = \{J_t\}_{0\le t\le 1}$, we consider the associated family of compatible metrics g_{J_t}. This family induces an L^2-metric on the space of paths on T^*N defined by

$$\langle\langle \xi_1, \xi_2\rangle\rangle_J = \int_0^1 g_{J_t}(\xi_1(t), \xi_2(t))dt = \int_0^1 \omega(\xi_1(t), J_t\xi_2(t))dt. \tag{12.2.7}$$

From now on, we will always denote by J a $[0, 1]$-*family* of compatible almost-complex structures unless stated otherwise.

Next we consider Hamiltonians $H = H(t, x)$ such that H_t is asymptotically constant, i.e., ones whose Hamiltonian vector field X_H is compactly supported. We define

$$\mathrm{supp}_{asc} H = \mathrm{supp}\, X_H := \bigcup_{t\in[0,1]} \mathrm{supp}\, X_{H_t}.$$

For each given compact set $K \subset T^*N$ and $R \in \mathbb{R}_+$, we define

$$\mathcal{H}_R = \mathcal{P}C^\infty_R(T^*N, \mathbb{R}) = \{H \in C^\infty([0, 1] \times T^*N, \mathbb{R}) \mid \mathrm{supp}_{asc} H \subset D^R(T^*N)\}, \tag{12.2.8}$$

which provides a natural filtration of the space $\mathcal{H}$. Then we have

$$\mathcal{H} := C^\infty([0, 1] \times T^*N, \mathbb{R}) = \bigcup_R \mathcal{H}_R$$

and equip the union $\cup_R\mathcal{H}_R$ with the direct limit topology of $\{\mathcal{H}_R\}_{R>0}$.

If we denote by $\mathrm{grad}_J\mathcal{A}_H$ the associated L^2-gradient vector field, the formula for $d\mathcal{A}_H(\gamma)$ and (12.1.4) imply that $\mathrm{grad}_J\mathcal{A}_H$ has the form

$$\mathrm{grad}_J\mathcal{A}_H(\gamma)(t) = J_t(\dot\gamma(t) - X_H(t, \gamma(t))), \tag{12.2.9}$$

which we simply write $J(\dot\gamma - X_H(\gamma))$. Therefore the *negative* gradient flow equation of a path $u : \mathbb{R} \to \Omega_{S_0S_1}$ has the form

$$\begin{cases} \partial u/\partial\tau + J\big(\partial u/\partial t - X_H(u)\big) = 0, \\ u(\tau, 0) \in \nu^*S_0,\ u(\tau, 1) \in \nu^*S_1, \end{cases} \tag{12.2.10}$$

if we regard u as a map $u : \mathbb{R} \times [0, 1] \to M$. We call this equation *Floer's perturbed Cauchy–Riemann equation* or simply the perturbed Cauchy–Riemann equation associated with the quadruple $(H, J; S_0, S_1)$.

The general Floer theory largely relies on the study of the moduli spaces of solutions $u : \mathbb{R} \times [0, 1] \to T^*N$ with *finite energy* and of *bounded image* of the kind (12.2.10) of perturbed Cauchy–Riemann equations. The relevant energy function is given by the following definition.

Definition 12.2.3 For a given smooth map $u : \mathbb{R} \times [0,1] \to M$, we define the energy, denoted by $E_{(H,J)}(u)$, of u by

$$E_{(H,J)}(u) = \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J_t}^2 + \left| \frac{\partial u}{\partial t} - X_H(u) \right|_{J_t}^2 \right) dt\, d\tau.$$

We denote by

$$\widetilde{\mathcal{M}}(H, J; S_0, S_1)$$

the set of bounded finite-energy solutions of (12.2.10) for general H not necessarily nondegenerate. The following lemma is an easy consequence of the condition of finite energy and bounded image.

Lemma 12.2.4 *Suppose that $J \in \mathcal{P}(\mathcal{J}^c)$ and H is any smooth Hamiltonian. If u satisfies*

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_H(u) \right) = 0$$

and has bounded image and finite energy, there exists a sequence $\tau_k \to \infty$ (respectively $\tau_k \to -\infty$) such that the path $z_k := u(\tau_k) = u(\tau_k, \cdot)$ converges in C^∞ to a solution $z : [0,1] \to T^*N$ of the Hamilton equation $\dot{x} = X_H(x)$.

Proof The proof of this lemma is similar to that of Lemma 11.2.2 and hence will be brief. Since u satisfies (12.2.10), we have $E_{(H,J)}(u) = \int |\partial u / \partial t - X_H(u)|_{J_t}^2 \, dt\, d\tau < \infty$. Therefore there exists a sequence $\tau_k \to \infty$ such that

$$\int_0^1 \left| \frac{\partial u}{\partial t}(\tau_k, \cdot) - X_H(u(\tau)) \right|_{J_t}^2 \to 0. \tag{12.2.11}$$

Denote $z_k := u(\tau_k, \cdot)$. *Since u is assumed to have a bounded image*, $|X_H(t, z_k(t))|$ is uniformly bounded over k. Then this boundedness and (12.2.11) imply that $\|z_k\|_{W^{1,2}} \le C$, with C independent of k. By Sobolev embedding $W^{1,2} \hookrightarrow C^\epsilon$ with $0 < \epsilon < \frac{1}{2}$, z_k is pre-compact in C^ϵ and so has a sequence converging to z_∞ in C^ϵ. From the equation $\dot{z}_k(t) = X_H(t, z_k(t))$, it follows that $\dot{z}_k(t)$ converges to $X_H(t, z_\infty(t))$ in C^ϵ. Now, by the same boot-strap argument as that of Lemma 11.2.2, we derive that z_k converges to z_∞ in C^∞. This finishes the proof. □

For $\widetilde{\mathcal{M}}(H, J; S_0, S_1)$ to be well-behaved when the parameter $(H, J; S_0, S_1)$ varies, we need to establish an a-priori energy bound and a C^0 bound for the map u satisfying (12.2.10). For this purpose, we need to impose a certain tameness of (H, J) at infinity. (We have mentioned the tameness condition on J before.)