Cambridge University Press
978-1-107-10732-8 — Quantum Gravity and the Functional Renormalization Group
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A Quantum Field Theory of Gravity

Today we know of four kinds of fundamental interactions which seem to underlie all elementary processes observed in nature. Three of them, the electromagnetic, the weak, and the strong interactions, are combined in the standard model of elementary particle physics, which has received striking experimental confirmation during the past decades. Regarded as a classical field theory, the model employs geometrically natural and mathematically well-understood structures such as connections of the Yang–Mills type, for example. Furthermore, being renormalizable in perturbation theory, we also know how the model can be elevated to the level of a perturbatively defined quantum field theory. Beyond this stage there are ongoing efforts directed toward a non-perturbative definition and evaluation of at least certain sectors of the theory. Here, modern concepts of statistical field theory have proven invaluable. They explain, for instance, how the renormalization properties of the original continuum theory are related to the behavior of appropriate statistical mechanics models on spacetime lattices when they approach a second-order phase transition. These insights opened the door for employing Monte-Carlo simulations as a non-perturbative tool in quantum field theory, and in particular, as a device to test for the "existence" of a theory beyond the realm of perturbation theory.

1.1 Renormalizing the Unrenormalizable

As for our theoretical understanding of the fourth of the fundamental interactions, gravity, the situation is markedly different from the other three forces of nature. With Einstein's general theory of relativity we have a classical field theory at our disposal which is spectacularly successful in explaining gravitational phenomena on scales that span many orders of magnitude. However, when we try to quantize General Relativity (GR) along the same lines as the standard model, we find that this road is blocked since the theory is non-renormalizable within perturbation theory [1, 2]. At higher orders of the loop expansion the calculations must cope with an increasing number of new types of divergences and they all must be absorbed by suitable counter terms. The finite parts of their coefficients are left undetermined by the theory itself and so they must be taken from experiment. While this does not exclude the possibility of computing quantum corrections to predictions of the classical theory, it implies that those corrections involve an increasing number of undetermined parameters as the perturbation order is increased.

As long as one restricts attention to a regime where only a few such new coupling constants play a role, quantized General Relativity (GR) has the status of an *effective quantum field theory* [3–5]. Similar to chiral perturbation theory [6] it makes unambiguous predictions for certain leading quantum corrections. With increasing energy increasingly high loop orders must be included and hence the predictions unavoidably involve a growing number of undetermined parameters. At this stage the theory gradually loses its predictive power, and ultimately it may break down completely. Increasing the order of perturbation theory beyond this point, would then have the paradoxical consequence of diminishing the theory's net predictive power as the hoped-for better precision is more than offset by the new undetermined parameters it introduces.

This loss of predictivity at high-energy or short-distance scales is a strong motivation to search for a *fundamental quantum theory* of gravity, i.e., a theory that is *predictive on all scales* and that admits *potentially large quantum effects*. Ideally this hypothetical theory would contain only a few free parameters whose values are not fixed by the theory itself. Similar to the familiar situation in perturbatively renormalizable models it would express all predictions as well-defined, computable functions of those few measured parameters.

Given that GR is not renormalizable in standard perturbation theory, it has commonly been argued that a satisfactory microscopic theory of the gravitational interaction cannot be set up within the realm of quantum field theory, at least not without adding further symmetries, extra dimensions, or new principles such as holography, for instance. By contrast, the Asymptotic Safety program retains quantum field theory without such additions to the theoretical arena. Instead, it abandons the traditional techniques of perturbation theory, the concepts of perturbative renormalization, and of perturbative renormalizability in particular.

The Asymptotic Safety approach is based on the generalized notion of renormalization shaped by Kadanoff and Wilson [7–9] and the use of a "functional" or "exact" renormalization group (RG) equation. Hence, concepts from modern statistical field theory play a pivotal role. They provide a unified framework for approaching the problems with both continuum and discrete methods.

In the new setting one can conceive of *non-perturbatively renormalizable* quantum field theories, i.e., models free from physically harmful divergences that remain predictive up to the highest energies even though they are non-renormalizable in the perturbative sense.

The idea that there might exist a non-perturbatively renormalizable, or as he called it, "asymptotically safe" quantum theory of gravity was first proposed by S. Weinberg in the late 1970s [10]. At that time no efficient tools to test this scenario were available. However, an epsilon-expansion valid near *two* dimensions indicated that the new path could indeed be viable, at least for an unphysical dimensionality of spacetime. Further encouragement came from certain matter field theories that likewise were not renormalizable within perturbation theory but could be shown to be non-perturbatively renormalizable. In a paper entitled "Renormalizing the Non-renormalizable," Gawedzki and Kupiainen [11] used a $1/N$-expansion to prove the non-perturbative renormalizability of the Gross–Neveu model in three dimensions [12, 13].

The systematic exploration of the Asymptotic Safety scenario in four dimensions began only in the 1990s when powerful functional renormalization group methods became available for the gravitational field [14]. The exposition of these non-perturbative methods and their use in scrutinizing the viability of the Asymptotic Safety route to a fundamental quantum field theory of gravity is the main topic of this book.

1.2 Background Independence

While the various approaches trying to unify the principles of quantum mechanics and General Relativity are based upon rather different physical ideas and are formulated in correspondingly different mathematical frameworks,[1] they all must cope with the problem of Background Independence in one way or another. Whatever the ultimate theory of quantum gravity, a central requirement we impose on it is that it should be Background Independent in the same sense as GR. Loosely speaking, this means the spacetime structure that is actually realized in nature should not be part of the theory's definition but rather arise as a solution to certain dynamical equations [18–20].

In classical General Relativity the spacetime structure is encoded in a Lorentzian metric on a smooth manifold, and this metric, via Einstein's equation, is a dynamical consequence of the matter present in the universe. In quantum gravity we would like to retain the fundamental idea that "matter tells space how to curve, and space tells matter how to move," but describe both "matter" and "space" quantum mechanically. While, today, it is fairly well understood how to set up quantum field theories of matter systems, the open key problem is the quantum mechanical description of "space."

In this book we will mostly explore the possibility of constructing a quantum field theory of gravity in which the spacetime metric carries the dynamical degrees of freedom which we associate to "space." Even though this property is taken

[1] For reviews of the attempts to obtain a quantum theory of gravitation see, for example, [15–17].

from GR, the fundamental dynamics of those metric degrees of freedom is allowed to be different from that in the classical theory.

The quantum theory of gravity we are searching for will be required to respect the following principle of Background Independence: In the formulation of the theory no special metric should play any distinguished role. The actual metric of spacetime should arise as the expectation value of the quantum field (operator) $\widehat{g}_{\mu\nu}$ with respect to some state: $g_{\mu\nu} = \langle \widehat{g}_{\mu\nu} \rangle$.

This requirement is in sharp contradistinction to the traditional setting of quantum field theory of matter systems on Minkowski space whose conceptual foundations heavily rely on the availability of a non-dynamical (rigid) Minkowski spacetime as a background structure.

The principle of Background Independence[2] can be rephrased more precisely as follows. We require that *none of the theory's basic rules and assumptions, and none of its predictions, therefore, may depend on any special metric that has been fixed a priori. All metrics of physical relevance must result from the intrinsic quantum gravitational dynamics.*

A possible objection against this working definition [21] could be as follows: A theory can be made "Background Independent" in the above sense, but nevertheless has a distinguished rigid background if the latter arises as the unique solution to some field equation which is made part of the "basic rules." For instance, rather than introducing a Minkowski background directly one instead imposes the field equation $R^{\mu}{}_{\nu\rho\sigma} = 0$. However, this objection can apply only in a setting where the dynamics, the field equations, can be chosen freely. In asymptotically safe gravity this is impossible since, as we shall see, the dynamical laws are dictated by the fixed-point action. They are thus a prediction rather than an input.

If we try to set up a continuum quantum field theory for the metric itself, even assuming we are given some plausible candidate for a microscopic dynamics, described by, say, a diffeomorphism invariant bare action functional S, then already well before one encounters the notorious problems related to the UV divergences, profound conceptual problems arise. Just to name one, in absence of a rigid background when the metric is dynamical, there is no preferred time direction, for instance, hence no notion of equal time commutators, and clearly the usual rules of quantization cannot be applied straightforwardly.

Many more problems arise when one tries to apply the familiar concepts and calculational methods of quantum field theory to the metric itself without introducing a rigid background structure. Some of them are conceptually deep while others are of a more technical nature.

The problems are particularly severe if one demands that the sought-for theory can also describe potential phases of gravity in which $\langle \widehat{g}_{\mu\nu} \rangle$ is degenerate

[2] Here, and in the following, we write "Background Independence" with capital letters when we refer to this principle rather than simply to the independence of some quantity with respect to the background field.

(non-invertible) or completely vanishing in the most extreme case. Interpreting $\langle \widehat{g}_{\mu\nu} \rangle$ as an order parameter analogous to the magnetization in a magnetic system, a non-degenerate classical metric $\langle \widehat{g}_{\mu\nu} \rangle$ would signal a spontaneous breaking of diffeomorphism invariance that leaves only the stability group of $\langle \widehat{g}_{\mu\nu} \rangle$ unbroken, i.e., the Poincaré group for example, when $\langle \widehat{g}_{\mu\nu} \rangle$ is given by the Minkowski metric. Conversely, $\langle \widehat{g}_{\mu\nu} \rangle \equiv 0$ would then be the hallmark of a phase with completely unbroken diffeomorphism invariance.

The analogy to magnetic systems suggests that this "unbroken phase" is much easier to deal with than those with $\langle \widehat{g}_{\mu\nu} \rangle \neq 0$. However, in practice this is not the case, and again the reason is that the traditional toolbox of quantum field theory as shaped by the requirements of particle or condensed matter physics has very little to offer as soon as $g_{\mu\nu}$ vanishes. The familiar actions for matter fields such as, say, $\int \sqrt{g} g^{\mu\nu} D_\mu \phi D_\nu \phi$ or $\int \sqrt{g} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}$, can no longer be written down since they require an *invertible* $g_{\mu\nu}$, and problems of this kind are clearly only the tip of the iceberg.

A similar difficulty shows up when one tries to conceive an appropriate notion of a "functional renormalization group" in the realm of quantum gravity. In standard field theory on a rigid background spacetime typical regularization schemes (by higher derivative regulators, for example) which are used to make the calculations well defined both in the infrared (IR) and the ultraviolet (UV) make essential use of the metric provided by this background spacetime. As a result, it is not obvious whether and how such schemes can carry over to quantum gravity.

This problem is particularly acute for non-perturbative approaches employing any kind of functional renormalization group equation (FRGE) that would implement a Wilson-like or "exact" renormalization group flow by a repeated coarse graining [22–35].

In conventional Euclidean field theory as it is employed in statistical mechanics, for instance, every such coarse-graining step comes equipped with an associated length scale. In the case of, say, block-spin transformations it measures the size of the spacetime blocks within which the microscopic degrees of freedom were averaged. But when the metric is dynamical and no rigid background is available, this concept becomes highly problematic since it is not clear in terms of which metric one should measure the physical, i.e. proper extension of a given spacetime block.

From a continuum viewpoint, in one way or another all techniques of functional renormalization involve a mode decomposition of the (field) configurations that are summed or integrated over in the partition function or functional integral. In the standard case the modes are often taken to be plane waves, characterized by a momentum vector p_μ. They should be thought of as the eigenfunctions of the Laplacian $\delta^{\mu\nu}\partial_\mu\partial_\nu$. These modes are grouped into two classes then, namely long wavelength (or IR) modes, and short-wavelength (or UV) modes, respectively, depending on whether the Euclidean magnitude of their momentum, $(\delta^{\mu\nu} p_\mu p_\nu)^{1/2}$, is smaller or bigger than a certain value. The short-wavelength modes are then "integrated out" and the resulting effective dynamics of the long-wavelength modes is deduced.

In the absence of an intrinsically given metric comparable to $\delta^{\mu\nu}$ this procedure fails for (at least) the following obvious reasons: There is neither a natural, physically motivated way of choosing the basis of field modes, nor is it clear how to discriminate between IR and UV modes and to fix the order in which the individual modes belonging to some (ad hoc) basis of field modes should be integrated out.

Thus, the potential danger one faces in applying the ideas of coarse graining and RG flows to a continuum formulation of gravity is that in the absence of a naturally provided metric there is a considerable degree of arbitrariness in the flow that might ruin the power the exact RG has otherwise. After all, most of its celebrated successes on both the foundational or conceptual side (understanding the nature of continuum limits, etc.) and the practical side (computing useful effective descriptions of a given fundamental theory) heavily rely on the rule "short wavelengths first, long wavelengths second" when it comes to integrating out degrees of freedom. Trivial as it sounds, nothing the like of it is available in a manifestly Background Independent continuum formulation of gravity.

1.3 All Backgrounds Is No Background

There are two quite different strategies for complying with the requirement of Background Independence:

(i) One can try to define the theory and work out its implications without ever employing a background metric or a similar non-dynamical structure. This is the path taken in Loop Quantum Gravity [36–39] and the discrete approaches to quantum gravity [40–48], for instance, where manifest Background Independence has dramatic consequences for the structure of the theory [49]. As we saw, it seems very hard, if not impossible, to realize it in a continuum field theory, however.[3]

(ii) One takes advantage of an arbitrary classical background metric $\bar{g}_{\mu\nu}$ at the intermediate steps of the quantization, but verifies at the end that no physical prediction depends on which metric was chosen. This *background field method* is at the heart of the continuum-based gravitational average action approach [14], which we shall employ in our investigation of asymptotic safety.

[3] Some of the difficulties are reminiscent of those encountered in the quantization of topological Yang–Mills theories. Even when the classical action can be written down without the need of a metric, the gauge fixing and quantization of the theory usually requires one. Hence, the only way of proving the topological character of some result is to show its independence of the metric chosen.

The two strategies have complementary advantages and disadvantages. Following the path (i), Background Independence is implemented strictly. Hence, it is manifest at all intermediate steps of the constructions and calculations, but one must then cope with the above profound difficulties. Taking the path (ii) instead, Background Independence is not manifest during the intermediate steps and requires an effort to reestablish it at the end. This strategy has the invaluable benefit that basically the entire arsenal of general concepts and technical tools of conventional background-dependent quantum field theory is applicable.

In the simplest variant of the background field method one parameterizes the quantum metric as $\widehat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ or a non-linear generalization thereof, and then quantizes the fluctuation $\widehat{h}_{\mu\nu}$ in essentially the same way one would quantize a matter field in a classical spacetime with metric $\bar{g}_{\mu\nu}$. In this way, all of the conceptual problems alluded to above, in particular the difficulties related to the construction of regulators, disappear.

Technically the quantization of gravity proceeds then almost as in standard field theory on a rigid classical spacetime, with one essential difference, though: In the latter, one concretely fixes the background $\bar{g}_{\mu\nu}$ typically as $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ or as $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ in the Euclidean case. In "background independent" quantum gravity instead, the metric $\bar{g}_{\mu\nu}$ is never specified concretely. All objects that one has to compute in this setting, generating functionals, say, are functionals of the variable $\bar{g}_{\mu\nu}$.

An example is the effective action $\Gamma[h_{\mu\nu}; \bar{g}_{\mu\nu}]$ that generates the dynamical equations for the expectation value $h_{\mu\nu} \equiv \langle \widehat{h}_{\mu\nu} \rangle$ of $\widehat{h}_{\mu\nu}$ and its higher n-point functions. It depends on both the background metric and the fluctuation expectation value. Similarly all n-point correlation functions of $\widehat{h}_{\mu\nu}$ which one computes from it have a parametric dependence on $\bar{g}_{\mu\nu}$. To stress this fact we sometimes write $h_{\mu\nu}[\bar{g}] \equiv \langle \widehat{h}_{\mu\nu} \rangle_{\bar{g}}$ for the 1-point function, for example.

Thus, in a sense, *the Background Independent quantization of gravity amounts to its quantization on all possible backgrounds simultaneously.*

There are now two metrics in the game which are equally important: the background $\bar{g}_{\mu\nu}$ and the expectation value of the full metric,

$$g_{\mu\nu} \equiv \langle \widehat{g}_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + h_{\mu\nu} \, , \qquad h_{\mu\nu} \equiv \langle \widehat{h}_{\mu\nu} \rangle. \tag{1.1}$$

Alternatively we may regard the effective action as a functional of the two metrics rather than $h_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. We define

$$\Gamma[g_{\mu\nu}, \bar{g}_{\mu\nu}] \equiv \Gamma[h_{\mu\nu}; \bar{g}_{\mu\nu}] \Big|_{h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}} . \tag{1.2}$$

Because of the almost symmetric status enjoyed by the two metrics we refer to this setting as the *"bi-metric" approach to the Background Independence problem.*

As for the notion of an "exact renormalization group" in quantum gravity, we will introduce the Effective Average Action (EAA) as a scale-dependent version of the ordinary effective action with a built-in IR cutoff at a variable mass scale

k and derive in particular a functional RG equation for it. As we will explain in more detail below, the construction of the EAA and its RG equation are only possible due to the presence of the classical background spacetime.

Despite the unavoidable bi-metric appearance of the background field method, the expectation value of the microscopic metric, $g_{\mu\nu}$, and the variable background metric $\bar{g}_{\mu\nu}$, enter physical quantities (observables) not independently but instead are constrained by a symmetry requirement. Obviously the full metric $\widehat{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ is invariant under the *split symmetry transformation* $\boldsymbol{\delta}\widehat{h}_{\mu\nu} = \varepsilon_{\mu\nu}$, $\boldsymbol{\delta}\bar{g}_{\mu\nu} = -\varepsilon_{\mu\nu}$ with an arbitrary symmetric tensor field $\varepsilon_{\mu\nu}$. At the quantum level, this transformation implies Ward identities for the n-point functions and the effective (average) action similar to those implied by gauge or Becchi–Rouet–Stora–Tyutin (BRST) invariance. In either case one must make sure that in the end the quantum theory constructed actually satisfies these Ward identities.

In a way, this is the point where one is paying the price for the many advantages the background field technique brings about. However, it will become clear that while the extra work necessary to implement split symmetry, and thus Background Independence at the quantum level, is a hard technical challenge, it does not involve insoluble problems of principle.

1.4 Asymptotic Safety in a Nutshell

In this section we give a concise preview of what Asymptotic Safety is about. Technical details and refinements are omitted as much as possible. They will be delivered later on in this book.

(1) The problem. The ultimate goal of the Asymptotic Safety program consists in giving a mathematically precise meaning to, and actually compute, functional integrals over "all" spacetime metrics of the form $\int \mathcal{D}\widehat{g}_{\mu\nu} \exp\big(iS[\widehat{g}_{\mu\nu}]\big)$, or

$$Z = \int \mathcal{D}\widehat{g}_{\mu\nu}\, e^{-S[\widehat{g}_{\mu\nu}]}\,, \tag{1.3}$$

from which all quantities of physical interest can be deduced then. Here $S[\widehat{g}_{\mu\nu}]$ denotes the classical or, more correctly, the bare action. It is required to be diffeomorphism invariant, but is kept completely arbitrary otherwise. In general it differs from the usual Einstein–Hilbert action. This generality is essential in the Asymptotic Safety program: the viewpoint is that the functional integral would exist only for a certain class of actions S and the task is to identify this class.

(2) The problem, reformulated. Following the approach proposed in [14] one attacks this problem in an indirect way: rather than dealing with the integral per se, one interprets it as the solution of a certain differential equation, a functional renormalization group equation, or "FRGE". The advantage is that, contrary to the functional integral, the FRGE is manifestly well defined. It can be seen as an "evolution equation" in a mathematical sense, defining an infinite dimensional dynamical system in which the RG scale plays the role of time.

Loosely speaking, this reformulation replaces the problem of defining functional integrals by the task of finding evolution histories of the dynamical system that extend to *infinitely late times*. According to the Asymptotic Safety conjecture the dynamical system possesses a fixed point which is approached at late times, yielding well-defined, fully extended evolutions, which in turn tell us how to construct (or "renormalize") the functional integral.

(3) From the functional integral to the FRGE. Let us now be slightly more explicit about the passage from the functional integral to the FRGE.

(3a) Formal character of the integral. Recall that in trying to put the purely formal functional integrals on a solid basis one is confronted with a number of obstacles:

(i) As in every field theory, difficulties arise since one tries to quantize *infinitely many degrees of freedom*. Therefore, at the intermediate steps of the construction one keeps only finitely many of them by introducing cutoffs at very small and very large distances, Λ^{-1} and k^{-1}, respectively. We shall specify their concrete implementation in a moment. The ultraviolet and infrared cutoff scales Λ and k, respectively, have the dimension of a mass, and the original system is recovered for $\Lambda \to \infty$, $k \to 0$.

(ii) We mentioned already that the most severe problem one encounters when trying to quantize gravity is the requirement of *Background Independence*. In the approach to Asymptotic Safety along the lines of [14] we follow the spirit of DeWitt's background field method [50, 51] and introduce a (classical, non-dynamical) background metric $\bar{g}_{\mu\nu}$, which is kept arbitrary. We then decompose the integration variable as $\widehat{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$, or a non-linear generalization thereof, and replace $\mathcal{D}\widehat{g}_{\mu\nu}$ with an integration over the fluctuation, $\mathcal{D}\widehat{h}_{\mu\nu}$. In this way one arrives at a conceptually easier task, namely the quantization of the matter-like field $\widehat{h}_{\mu\nu}$ in a generic, but classical background $\bar{g}_{\mu\nu}$.

The availability of the background metric is crucial at various stages of the construction of an FRGE. However, the final physical results do not depend on the choice of a specific background.

(iii) As in every gauge theory, the *redundancy of gauge-equivalent field configurations* (diffeomorphic metrics) has to be carefully accounted for. Here we employ the Faddeev–Popov method and add a gauge-fixing term $S_{\mathrm{gf}} \propto \int \sqrt{\bar{g}}\bar{g}^{\mu\nu} F_\mu F_\nu$ to S where $F_\mu \equiv F_\mu(\widehat{h};\bar{g})$ is chosen such that the condition $F_\mu = 0$ picks a single representative from each gauge orbit. The resulting volume element on orbit space, the Faddeev–Popov determinant, is expressed as a functional integral over Grassmannian ghost fields C^μ and $\bar{C}_\mu$, governed by an action S_{gh}.

In this way (1.3) gets replaced by $\widetilde{Z}[\bar{\Phi}] = \int \mathcal{D}\widehat{\Phi}\, \exp\big(-S_{\rm tot}[\widehat{\Phi}, \bar{\Phi}]\big)$. Here the total bare action $S_{\rm tot} \equiv S + S_{\rm gf} + S_{\rm gh}$ depends on the dynamical fields $\widehat{\Phi} \equiv (\widehat{h}_{\mu\nu}, C^\mu, \bar{C}_\mu)$, the background fields $\bar{\Phi} \equiv (\bar{g}_{\mu\nu})$, and possibly also on (both dynamical and background) matter fields, which for simplicity are not included here.

(3b) Standard effective action. Using the gauge fixed and regularized integral we can compute arbitrary ($\bar{\Phi}$-dependent!) expectation values $\langle \mathcal{O}(\widehat{\Phi})\rangle \equiv \widetilde{Z}^{-1} \int \mathcal{D}\widehat{\Phi}\, \mathcal{O}(\widehat{\Phi})\, e^{-S_{\rm tot}[\widehat{\Phi}, \bar{\Phi}]}$; for instance, n-point functions where $\mathcal{O}$ consists of strings $\widehat{\Phi}(x_1)\widehat{\Phi}(x_2)\ldots\widehat{\Phi}(x_n)$. For $n = 1$ we use the notation $\Phi \equiv \langle\widehat{\Phi}\rangle \equiv (h_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu)$, i.e., the elementary field expectation values are $h_{\mu\nu} \equiv \langle\widehat{h}_{\mu\nu}\rangle$, $\xi^\mu \equiv \langle C^\mu\rangle$ and $\bar{\xi}_\mu \equiv \langle\bar{C}_\mu\rangle$. Thus, the full dynamical metric has the expectation value $g_{\mu\nu} \equiv \langle\widehat{g}_{\mu\nu}\rangle = \bar{g}_{\mu\nu} + h_{\mu\nu}$.

The dynamical laws which govern the expectation value $\Phi(x)$ have an elegant description in terms of the *effective action* Γ. It is a functional depending on Φ similar to the classical $S[\Phi]$ to which it reduces in the classical limit. Requiring stationarity, S yields the classical field equation $(\delta S/\delta\Phi)[\Phi_{\rm class}] = 0$, while Γ gives rise to a quantum mechanical analog satisfied by the expectation values, the *effective field equation* $(\delta\Gamma/\delta\Phi)[\langle\widehat{\Phi}\rangle] = 0$.

If, as in the case at hand, $\Gamma \equiv \Gamma[\Phi, \bar{\Phi}] \equiv \Gamma[h_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu; \bar{g}_{\mu\nu}]$ also depends on background fields, the solutions of this equation inherit this dependence and thus $h_{\mu\nu} \equiv \langle\widehat{h}_{\mu\nu}\rangle$ functionally depends on $\bar{g}_{\mu\nu}$.

Technically, Γ is obtained from a functional integral with $S_{\rm tot}$ replaced by $S^J_{\rm tot} \equiv S_{\rm tot} - \int {\rm d}x\, J(x)\widehat{\Phi}(x)$. The new term couples the dynamical fields to external, classical sources $J(x)$ and repeated functional differentiation $(\delta/\delta J)^n$ of $\ln \widetilde{Z}[J, \bar{\Phi}]$ yields the n-point functions. In particular, $\Phi = \delta \ln \widetilde{Z}/\delta J$. It is a standard result that $\Gamma[\Phi, \bar{\Phi}]$ equals exactly the Legendre transform of $\ln \widetilde{Z}[J, \bar{\Phi}]$, at fixed background fields $\bar{\Phi}$.

The importance of Γ also resides in the fact that *it is the generating functional of special n-point functions from which all others can be easily reconstructed.* Therefore, finding Γ in a given quantum field theory is often considered equivalent to completely "solving" the theory.

(3c) Notions of gauge invariance. In practical applications of $\Gamma[\Phi, \bar{\Phi}]$ it is advantageous to employ a gauge-breaking condition F_μ that fixes a gauge belonging to the distinguished class of the so-called *background gauges.* To see the benefit, recall that the original gauge transformations read $\delta\widehat{g}_{\mu\nu} = \mathcal{L}_v \widehat{g}_{\mu\nu}$ where $\mathcal{L}_v$, denotes the Lie derivative with regards to the vector field v.

When we decompose $\widehat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ we can distribute the gauge variation of $\widehat{g}_{\mu\nu}$ in different ways over $\bar{g}_{\mu\nu}$ and $\widehat{h}_{\mu\nu}$. In particular, this gives rise to what is known as *quantum gauge transformations*,

$$\boldsymbol{\delta}^{\mathbf{Q}}\widehat{h}_{\mu\nu} = \mathcal{L}_v(\bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu})\,, \quad \boldsymbol{\delta}^{\mathbf{Q}}\bar{g}_{\mu\nu} = 0 \tag{1.4}$$