1

Models for Homotopy Theories

In this chapter, we introduce the main ideas of modeling homotopy theories. Since the main objective of this book is to understand the homotopy theory of $(\infty, 1)$-categories, this material allows us to put this idea into a rigorous framework. Most significantly, we explain the structure of a model category, as developed by Quillen. However, most of the material here is to be regarded as background, so very few proofs are given; we give numerous references, so that a reader unfamiliar with certain concepts can find more details elsewhere.

1.1 Some Basics in Category Theory

We begin with a brief review of some essential definitions in category theory.

Definition 1.1.1 A *category* C consists of:

- a collection of *objects*, $\mathrm{ob}(C)$, and
- for any $x, y \in \mathrm{ob}(C)$, a set of *morphisms*, denoted $\mathrm{Hom}_C(x, y)$, such that
- if $f \in \mathrm{Hom}_C(x, y)$ and $g \in \mathrm{Hom}_C(y, z)$, then there is a *composite* morphism $g \circ f \in \mathrm{Hom}_C(x, z)$, and
- given any object x in C, there is an identity morphism $\mathrm{id}_x \in \mathrm{Hom}_C(x, x)$.

If $f \in \mathrm{Hom}_C(x, y)$, we often write $f : x \to y$, and say that x is the *source* of f and that y is the *target* of f.

These data are required to satisfy the following two axioms.

- (Associativity) If $f : w \to x$, $g : x \to y$, and $h : y \to z$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (Unit) Given any $f: x \to y$, we have

$$f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f.$$

Example 1.1.2 The category of sets, denoted by $\mathcal{S}ets$, has as objects all sets and as morphisms all functions between sets.

Example 1.1.3 The category of groups, denoted by $\mathcal{G}ps$, has as objects all groups and as morphisms all group homomorphisms.

Definition 1.1.4 A category $\mathcal{C}$ is *small* if $\mathrm{ob}(\mathcal{C})$ is a set.

Example 1.1.5 Let $n \geq 0$ be a natural number. Consider the category $[n]$ with objects $0, 1, \ldots, n$ and morphisms defined by

$$\mathrm{Hom}_{[n]}(i, j) = \begin{cases} * & i \leq j \\ \varnothing & i > j. \end{cases}$$

Here, by $*$ we mean a one-element set. Then $[n]$ is an example of a small category. We can depict $[n]$ as

$$0 \to 1 \to 2 \to \cdots \to n.$$

Observe, in contrast, that neither the category of groups nor the category of sets is small.

In a category, we often distingush the morphisms which are invertible.

Definition 1.1.6 A morphism $f: x \to y$ in a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g: y \to x$ such that $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$. A category $\mathcal{C}$ is a *groupoid* if all its morphisms are isomorphisms.

We also note the following special kinds of objects that a category might possess.

Definition 1.1.7 An object $\varnothing$ of a category $\mathcal{C}$ is *initial* if, for any object c in $\mathcal{C}$, there is a unique morphism $\varnothing \to c$ in $\mathcal{C}$. Dually, an object $*$ is *terminal* if, for any object c of $\mathcal{C}$, there is a unique morphism $c \to *$ in $\mathcal{C}$. If an object is both initial and terminal, it is called a *zero object*.

Proposition 1.1.8 [4, 2.8] *Initial and terminal objects in a category are unique up to isomorphism.*

Let us look at a few ways to obtain new categories from ones we already have. One basic way is to reverse the direction of the morphisms.

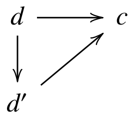
Definition 1.1.9 Let C be a category. Its *opposite category* is the category C^{op} with the same objects as C and morphisms defined by

$$\mathrm{Hom}_{C^{op}}(x, y) = \mathrm{Hom}_C(y, x).$$

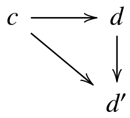
Definition 1.1.10 A *subcategory* $\mathcal{D}$ of a category C consists of a subclass of the objects of C and, for any objects x and y, a subset $\mathrm{Hom}_{\mathcal{D}}(x, y) \subseteq \mathrm{Hom}_C(x, y)$, such that $\mathcal{D}$ also satisfies the necessary conditions to be a category.

Definition 1.1.11 Let C be a category. A *full subcategory* of C is a category $\mathcal{D}$ whose objects form a subclass of the objects of C and for which $\mathrm{Hom}_{\mathcal{D}}(c, c') = \mathrm{Hom}_C(c, c')$.

Definition 1.1.12 Let C be a category and c an object of C. The category of *objects of C over c* has objects given by morphisms $d \to c$ in C and morphisms the maps $d \to d'$ in C making the diagram



commute. This category is denoted by $C \downarrow c$ or by C/c. Dually, the category of *objects of C under c* has objects given by morphisms $c \to d$ and morphisms the maps $d \to d'$ making the diagram



commute. This category is denoted by $c \downarrow C$.

We can also consider functions between categories.

Definition 1.1.13 Let C and $\mathcal{D}$ be categories. A *functor* $F : C \to \mathcal{D}$ assigns to any object x of C an object $F(x)$ of $\mathcal{D}$, and to any morphism $f : x \to y$ of C a morphism $F(f)$ of $\mathcal{D}$, such that

- $F(f) : F(x) \to F(y)$,
- $F(g \circ f) = F(g) \circ F(f)$, and
- $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ for every object x of C.

Example 1.1.14 The collection of small categories and functors between them itself forms a category, which we denote by $\mathcal{C}at$.

We might ask whether two categories are essentially the same. While we could demand that categories only be considered equivalent if their objects and morphisms are in bijection with one another, we typically consider instead the following definition.

Definition 1.1.15 A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if:

1 for every object x and y of $\mathcal{C}$, the map of sets $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$ is an isomorphism, and
2 F is essentially surjective, i.e., for any object d of $\mathcal{D}$, there exists an object c in $\mathcal{C}$ together with an isomorphism $F(c) \to d$ in $\mathcal{D}$.

In this case, we say the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*.

If a functor F satisfies the first condition for equivalence of categories, it is said to be *fully faithful*. It is *full* if each such map is surjective and *faithful* if each such map is injective.

We are often interested not just in a functor from one category to another, but in pairs of functors which go back and forth between two categories in a suitably compatible way.

Definition 1.1.16 Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are functors. The pair (F,G) is an *adjoint pair* of functors if, for any object x of $\mathcal{C}$ and object y of $\mathcal{D}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(x), y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G(y)).$$

The functor F is called the *left adjoint* and the functor G is called the *right adjoint*. We often write an adjoint pair as

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$$

and employ the convention that the left adjoint always appears as the topmost arrow.

Example 1.1.17 There is a forgetful functor $\mathcal{G}ps \to \mathcal{S}ets$ which takes a group to its underlying set. This functor has a left adjoint, taking a set to the free group on that set. Such an adjoint pair is called a *forgetful-free adjunction*.

A functor from a small category $\mathcal{D}$ to an arbitrary category $\mathcal{C}$ can be thought of as picking out a configuration of objects and morphisms in $\mathcal{C}$ which have the shape of $\mathcal{D}$.

Definition 1.1.18 Let $\mathcal{C}$ be a category and $\mathcal{D}$ a small category. A $\mathcal{D}$*-diagram* in $\mathcal{C}$ is a functor $\mathcal{D} \to \mathcal{C}$.

Within a category, we are often interested in objects which satisfy certain universal properties with respect to diagrams in that category. Hence, we turn to limits and colimits.

Definition 1.1.19 Let $D\colon \mathcal{D} \to \mathcal{C}$ be a diagram. A *limit* for D is an object $\lim_{\mathcal{D}} D$ of $\mathcal{C}$ such that there are maps $\lim_{\mathcal{D}} D \to D(d)$ for every object d of $\mathcal{D}$, compatible in the sense that, if $d \to e$ is a morphism in $\mathcal{D}$, there is a commutative triangle

$$\begin{array}{ccc} \lim_{\mathcal{D}} D & \longrightarrow & D(d) \\ & \searrow & \downarrow \\ & & D(e), \end{array}$$

and these triangles are all compatible with one another. Furthermore, the object $\lim_{\mathcal{D}} D$ is universal in the sense that, if there exists any other object c of $\mathcal{C}$ together with such maps, then each map $c \to D(d)$ factors through $\lim_{\mathcal{D}} D$.

In particular, if a limit of a diagram exists, it is unique up to unique isomorphism.

Definition 1.1.20 A category $\mathcal{C}$ *has all small limits* if, for every diagram $D\colon \mathcal{D} \to \mathcal{C}$, with $\mathcal{D}$ small, the limit $\lim_{\mathcal{D}} D$ exists.

We can similarly define what it means for a category to have all finite limits. We now give a few of the most common examples of limits.

Definition 1.1.21 A *product* is a limit of a diagram consisting of objects but no nonidentity morphisms. A *pullback* is a limit of a diagram of the form

$$(\bullet \to \bullet \leftarrow \bullet)\,.$$

An *equalizer* is a limit of a diagram of the form

$$(\bullet \rightrightarrows \bullet)\,.$$

We give two criteria for determining whether certain kinds of limits exist in a category.

Proposition 1.1.22 [4, 5.23] *If a category has pullbacks and a terminal object, then it has all finite limits.*

Proposition 1.1.23 [4, 5.24] *If a category has all small products and all equalizers, then it has all small limits.*

We similarly have the dual notion of colimit of a diagram.

Definition 1.1.24 Let $D\colon \mathcal{D} \to \mathcal{C}$ be a diagram. A *colimit* for D is an object $\operatorname{colim}_{\mathcal{D}} D$ of $\mathcal{C}$ such that there are maps $D(d) \to \operatorname{colim}_{\mathcal{D}} D$ for every object d of $\mathcal{D}$, compatible in the sense that, if $d \to e$ is a morphism in $\mathcal{D}$, there is a commutative triangle

$$\begin{array}{ccc} D(d) & \longrightarrow & \operatorname{colim}_{\mathcal{D}} D \\ \downarrow & \nearrow & \\ D(e) & & \end{array},$$

and these triangles are all compatible with one another. Furthermore, the object $\operatorname{colim}_{\mathcal{D}} D$ is universal in the sense that, if there exists any other object c of $\mathcal{C}$ together with such maps, then each map $D(d) \to c$ factors through $\operatorname{colim}_{\mathcal{D}} D$.

Again, if colimits exist, then they are unique up to unique isomorphism.

Definition 1.1.25 A category $\mathcal{C}$ *has all small colimits* if, for every diagram $D\colon \mathcal{D} \to \mathcal{C}$, with $\mathcal{D}$ small, the colimit $\operatorname{colim}_{\mathcal{D}} D$ exists.

Definition 1.1.26 A *coproduct* is a colimit of a diagram consisting of objects but no nonidentity morphisms. A *pushout* is a colimit of a diagram of the form

$$(\bullet \leftarrow \bullet \rightarrow \bullet).$$

A *coequalizer* is a colimit of a diagram of the form

$$(\bullet \rightrightarrows \bullet).$$

We will have need of the following kinds of colimits as well.

Definition 1.1.27 [90, IX.1] A nonempty category $\mathcal{D}$ is *filtered* if

1 for any two objects d and d' of $\mathcal{D}$, there exists an object e together with morphisms $d \to e$ and $d' \to e$, and
2 given two different morphisms $u, v\colon c \to d$, there exist an object e and morphism $w\colon d \to e$ such that $wu = wv$.

If $F\colon \mathcal{D} \to \mathcal{C}$ is a functor with $\mathcal{D}$ a filtered category, then the colimit of F is called a *filtered colimit*. If $\mathcal{D}$ is a partially ordered set (so that there is only one possible morphism $i \to j$ in $\mathcal{D}$) which satisfies condition (1), then $\mathcal{D}$ is a *directed* poset and a colimit of a functor $F\colon \mathcal{D} \to \mathcal{C}$ is a *directed colimit*.

Similarly to the case for limits, we have criteria for when a category has certain kinds of colimits.

Proposition 1.1.28 [4, 5.25] *If a category has all small coproducts and all coequalizers, then it has all small colimits.*

Proposition 1.1.29 [4, 9.14] *Left adjoint functors preserve colimits and right adjoint functors preserve limits.*

Remark 1.1.30 We can define initial and terminal objects in terms of limits and colimits. Consider the empty category $\varnothing$ with no objects. For any category $\mathcal{C}$, the limit of the functor $\varnothing \to \mathcal{C}$ (if it exists) is a terminal object of $\mathcal{C}$; and the colimit of such a functor (if it exists) is an initial object of $\mathcal{C}$.

Not only do we have functors between categories, but we can also consider interactions between different functors.

Definition 1.1.31 Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta \colon F \to G$ is a family of morphisms $\eta_c \colon F(c) \to G(c)$ in $\mathcal{D}$, indexed over all objects c of $\mathcal{C}$, such that, for any morphism $f \colon c \to c'$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\eta_{c'}} & G(c') \end{array}$$

commutes. A *natural isomorphism* is a natural transformation η such that each morphism η_c is an isomorphism in $\mathcal{D}$.

Given this definition, we can give an equivalent formulation of what it means to be an equivalence of categories.

Proposition 1.1.32 [4, 7.25] *A functor $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if there exists a functor $G \colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $GF \cong \mathrm{id}_{\mathcal{C}}$ and $FG \cong \mathrm{id}_{\mathcal{D}}$ to the respective identity functors.*

We can also use natural transformations to assemble the functors between two fixed categories into a category.

Example 1.1.33 Let $\mathcal{C}$ be a category and $\mathcal{D}$ a small category. There is a category of diagrams $\mathcal{C}^{\mathcal{D}}$ whose objects are functors $\mathcal{D} \to \mathcal{C}$ and whose morphisms are natural transformations.

If $\mathcal{D} = [1]$, the category depicted by

$$\bullet \to \bullet,$$

then $\mathcal{C}^{[1]}$ is the category whose objects are morphisms of $\mathcal{C}$.

Of particular interest are functors to the category of sets which are represented by an object, in the following sense.

Definition 1.1.34 Let $\mathcal{C}$ be a category. A functor $\mathcal{C} \to \mathcal{S}ets$ is *representable* if it is of the form $\mathrm{Hom}_{\mathcal{C}}(-, c)$ for some object c of $\mathcal{C}$.

Definition 1.1.35 Let $\mathcal{D}$ be a small category. The *Yoneda embedding* is the functor $y\colon \mathcal{D} \to \mathcal{S}ets^{\mathcal{D}^{op}}$ which sends an object d of $\mathcal{D}$ to the representable functor $\mathrm{Hom}_{\mathcal{D}}(-, d)$ and a morphism $f\colon d \to d'$ to the natural transformation $\mathrm{Hom}_{\mathcal{D}}(-, d) \to \mathrm{Hom}_{\mathcal{D}}(-, d')$.

The following result gives some indication of why we like representable functors.

Lemma 1.1.36 (Yoneda lemma) [4, 8.2] *Let $\mathcal{C}$ be a small category and $F\colon \mathcal{C}^{op} \to \mathcal{S}ets$ a functor. Given any object c of $\mathcal{C}$, there is an isomorphism $\mathrm{Hom}_{\mathcal{S}ets^{\mathcal{C}^{op}}}(y(c), F) \cong F(c)$ of sets which is natural in both F and c.*

Now we turn our attention to additional structures on categories. The first such structure has the additional data of a binary operation on the objects of a category.

Definition 1.1.37 A *monoidal category* is a category $\mathcal{C}$ which is equipped with

1 a tensor product functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, where the image of a pair of objects (x, y) is denoted by $x \otimes y$,
2 a *unit object* I,
3 for every $x, y, z \in \mathrm{ob}(\mathcal{C})$, an associativity isomorphism

$$a_{x,y,z}\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z),$$

natural in the objects x, y, and z, and
4 for every $x \in \mathrm{ob}(\mathcal{C})$, a left unit isomorphism $\ell_x\colon I \otimes x \to x$ and a right unit isomorphism $r_x\colon x \otimes I \to x$, both natural in x.

We further assume that the diagrams

$$\begin{array}{ccccc}
 & & ((w \otimes x) \otimes y) \otimes z & & \\
 & \swarrow^{a_{w,x,y} \otimes \mathrm{id}_z} & & \searrow^{a_{w \otimes x, y, z}} & \\
(w \otimes (x \otimes y)) \otimes z & & & & (w \otimes x) \otimes (y \otimes z) \\
\downarrow{\scriptstyle a_{w, x \otimes y, z}} & & & & \downarrow{\scriptstyle a_{w,x,y \otimes z}} \\
w \otimes ((x \otimes y) \otimes z) & & \xrightarrow{\mathrm{id}_w \otimes a_{x,y,z}} & & w \otimes (x \otimes (y \otimes z))
\end{array}$$

and

$$\begin{array}{ccc} (x \otimes I) \otimes z & \xrightarrow{a_{x,I,y}} & x \otimes (I \otimes y) \\ & {\scriptstyle r_x \otimes \mathrm{id}_y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_x \otimes \ell_y} & \\ & x \otimes y & \end{array}$$

commute for any objects w, x, y, and z of $\mathcal{C}$.

We denote such a monoidal category by $(\mathcal{C}, \otimes, I)$ when we want to emphasize the tensor product and unit.

Definition 1.1.38 A monoidal category $(\mathcal{C}, \otimes, I)$ is *symmetric* if, additionally, it is equipped with isomorphisms $s_{x,y} \colon x \otimes y \to y \otimes x$ for any objects x and y of $\mathcal{C}$, natural in x and y, such that the diagrams

$$\begin{array}{ccccc} & (x \otimes y) \otimes z & \xrightarrow{s_{x,y} \otimes \mathrm{id}_z} & (y \otimes x) \otimes z & \\ {\scriptstyle a_{x,y,z}} \swarrow & & & & \searrow {\scriptstyle a_{y,x,z}} \\ x \otimes (y \otimes z) & & & & y \otimes (x \otimes z) \\ {\scriptstyle s_{x,y \otimes z}} \searrow & & & & \swarrow {\scriptstyle \mathrm{id}_y \otimes s_{x,z}} \\ & (y \otimes z) \otimes x & \xrightarrow{a_{y,z,x}} & y \otimes (z \otimes x) & \end{array},$$

$$\begin{array}{ccc} x \otimes I & \xrightarrow{s_{x,I}} & I \otimes x \\ & {\scriptstyle r_x} \searrow \quad \swarrow {\scriptstyle \ell_x} & \\ & x & \end{array},$$

and

$$\begin{array}{ccc} x \otimes y & \xrightarrow{s_{x,y}} & y \otimes x \\ & {\scriptstyle id_{x \otimes y}} \searrow & \downarrow {\scriptstyle s_{y,x}} \\ & & x \otimes y \end{array}$$

commute for all objects x, y, and z of $\mathcal{C}$.

Definition 1.1.39 A symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is *closed* if, for any object y of $\mathcal{C}$, the functor $- \otimes y \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint.

The right adjoint is usually denoted by $(-)^y$; an object of $\mathcal{C}$ in the image of this functor is called an *internal hom object* of $\mathcal{C}$. One can also define what it means for a nonsymmetric model category to be closed.

Now we turn to the case where the hom sets in a category have extra structure.

Definition 1.1.40 Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A *category $\mathcal{D}$ enriched in $\mathcal{C}$* consists of

1. a collection of objects, denoted by $\mathrm{ob}(\mathcal{D})$,
2. for any pair $x, y \in \mathrm{ob}(\mathcal{D})$, an object $\mathrm{Map}_{\mathcal{D}}(x, y)$ of $\mathcal{C}$,
3. for every x, y, z in $\mathrm{ob}(\mathcal{D})$, a composition morphism
$$c_{x,y,z}\colon\ \mathrm{Map}_{\mathcal{D}}(x, y) \otimes \mathrm{Map}_{\mathcal{D}}(y, z) \to \mathrm{Map}_{\mathcal{D}}(x, z)$$
in $\mathcal{C}$, and
4. for every $x \in \mathrm{ob}(\mathcal{D})$, a unit map $\eta_x\colon I \to \mathrm{Map}_{\mathcal{D}}(x, x)$ such that the diagrams
$$\begin{array}{ccc} I \otimes \mathrm{Map}_{\mathcal{D}}(x, y) & \xrightarrow{\eta_x \otimes \mathrm{id}} & \mathrm{Map}_{\mathcal{D}}(x, x) \otimes \mathrm{Map}_{\mathcal{D}}(x, y) \\ & \searrow \quad \swarrow {\scriptstyle c_{x,x,y}} & \\ & \mathrm{Map}_{\mathcal{D}}(x, y) & \end{array}$$
and
$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}(w, x) \otimes I & \xrightarrow{\mathrm{id} \otimes \eta_x} & \mathrm{Map}_{\mathcal{D}}(w, x) \otimes \mathrm{Map}_{\mathcal{D}}(x, x) \\ & \searrow \quad \swarrow {\scriptstyle c_{w,x,x}} & \\ & \mathrm{Map}_{\mathcal{D}}(w, x) & \end{array}$$
are commutative.

We assume that composition is associative, in that, for any $w, x, y, z \in \mathrm{ob}(\mathcal{D})$, the diagram
$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}(w, x) \otimes \mathrm{Map}_{\mathcal{D}}(x, y) \otimes \mathrm{Map}_{\mathcal{D}}(y, z) & \xrightarrow{c_{w,x,y} \otimes \mathrm{id}} & \mathrm{Map}_{\mathcal{D}}(w, y) \otimes \mathrm{Map}_{\mathcal{D}}(y, z) \\ {\scriptstyle \mathrm{id} \otimes c_{x,y,z}} \downarrow & & \downarrow {\scriptstyle c_{w,y,z}} \\ \mathrm{Map}_{\mathcal{D}}(w, x) \otimes \mathrm{Map}_{\mathcal{D}}(x, z) & \xrightarrow{c_{w,x,z}} & \mathrm{Map}_{\mathcal{D}}(w, z) \end{array}$$
commutes.

1.2 Weak Equivalences and Localization

The main idea of homotopy theory is that a category $\mathcal{C}$ may have morphisms which are not isomorphisms, but which we would like to regard as equivalences. We begin with a few classical examples.