Singularities: Formation, Structure, and Propagation

Many key phenomena in physics and engineering can be understood as singularities in the solutions to the differential equations describing them. Examples covered thoroughly in this book include the formation of drops and bubbles, the propagation of a crack, and the formation of a shock in a gas.

Aimed at a broad audience, this book provides the mathematical tools for understanding singularities and explains the many common features in their mathematical structure. Part I introduces the main concepts and techniques, using the most elementary mathematics possible so that it can be followed by readers with only a general background in differential equations. Parts II and III require more specialized methods of partial differential equations, complex analysis, and asymptotic techniques. The book may be used for advanced fluid mechanics courses and as a complement to a general course on applied partial differential equations.

J. EGGERS is Professor of Applied Mathematics at the University of Bristol. His career has been devoted to the understanding of self-similar phenomena, and he has more than 15 years of experience in teaching nonlinear and scaling phenomena to undergraduate and postgraduate students. Eggers has made fundamental contributions to our mathematical understanding of free surface flows, in particular the breakup and coalescence of drops. His work was instrumental in establishing the study of singularities as a research field in applied mathematics and in fluid mechanics. He is a member of the Academy of Arts and Sciences in Erfurt, Germany, and a Fellow of the American Physical Society and has recently been made a Euromech Fellow.

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

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www.cambridge.org
Information on this title: www.cambridge.org/9781107098411

First published 2015

Printed in the United Kingdom by Bell and Bain Ltd

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication Data
Eggers, J. (Jens G.), 1963–
Singularities : formation, structure, and propagation / J. Eggers, University of Bristol, M.A. Fontelos, Consejo Superior de Investigaciones Cientificas, Madrid.
pages cm. – (Cambridge texts in applied mathematics)
Includes bibliographical references and index.
ISBN 978-1-107-09841-1
1. Singularities (Mathematics) I. Fontelos, M. A. (Marco Antonio) II. Title.
QA614.58.E325 2015
514′.746–dc23
2015016866

ISBN 978-1-107-09841-1 Hardback
ISBN 978-1-107-48549-5 Paperback

To our parents:

Gisela and Hans,
Blanca and Julian.

Contents

Preface

The word “singularity” is used popularly to describe exceptional events at which something changes radically or where a new structure emerges. In the mathematical language of this book, we speak of a singularity when some quantity goes to infinity. This is usually related to the solution of a differential equation which loses smoothness in that either the unknown itself or its derivatives become unbounded at some point or region of their domain.

Very often a singularity understood in the strict mathematical sense justifies the popular use of the word, since it represents a situation or structure of special interest. For example, a singularity of the curvature lies at the center of a black hole, which is formed after the collapse of a supermassive star, and the universe itself is generally believed to have begun at a singularity. Unfortunately, the real difficulty here lies with the correct physical interpretation of the mathematical solution, which is the reason we have not been able to include examples from general relativity.

Examples of singularities discussed in this book are vortices, such as the flow around the center of a tornado, shock waves generated by the motion of a supersonic plane, caustic lines of intense brightness produced by the focusing of light, and the formation of a drop that results from the discontinuous separation of a liquid mass into two or several pieces.

Starting in the nineteenth century with the study of shock waves, singularities have been investigated on an individual basis. They have remained one of the most exciting research topics in both pure and applied mathematics. For example, two of the seven Millennium Prize problems, proposed by the Clay Foundation, were directly or indirectly related to singularities. The sixth problem was to investigate whether the Navier–Stokes equation, which describes the motion of fluids, does or does not produce any singularities. A related and hotly debated problem poses the same question for the Euler equation, which is the Navier–Stokes equation in the absence of viscosity. Both problems are still to be solved.

The third Millennium Problem, known as the Poincaré conjecture, was solved by G. Y. Perelman while studying the singularities of the partial differential equation describing Ricci flow (similar equations will be studied in Chapter 9 of this book). Perelman used his insight into the structure of these singularities, by continuing the flow across the singularity in such a way that the essential topological information was preserved.

However, few attempts have been made to present a general survey of singularities which would bring out their unifying features. In our opinion, the most important shared feature is that of self-similarity, which runs as a common thread through this book and which will be highlighted in each individual case. The significance of self-similarity and scaling was also expounded in Barenblatt's influential book [14], although the focus was not on singularities.

Self-similarity was already embodied in the similarity solution introduced in 1934 by J. Leray [138] to construct singular solutions to the Navier–Stokes equation. This posits that the solution is invariant as a function of time (or some other physical parameter), up to a change of scale. The existence of a scaling symmetry implies a dimensional reduction of the problem and reduces it to the study of the neighborhood of the singularity. This greatly simplifies the mathematical problem and makes it amenable to explicit analytical calculation. In this book we will largely ignore the important problems of existence and uniqueness but, rather, will focus on obtaining explicit solutions which can be compared to experimental data and used to explain qualitative experimental features.

Our book is intended for a broad audience of students and scientists, mainly in the areas of mathematics, physics, and engineering. It is in three parts:

- setting the scene
- formation of singularities
- persistent singularities: propagation.

The first part introduces the main concepts using elementary mathematics that can be followed by undergraduate students in their final years. The only requirement is a basic knowledge of ordinary and simple linear partial differential equations. Our aim is to introduce the fundamental ideas of blowup, self-similarity, and regularization, and to provide some essential mathematical tools such as asymptotics and matching. We introduce (or remind advanced readers of) the main concepts of continuum mechanics and develop two important tools in the study of singularities: local singular expansions and asymptotic expansions of partial differential equations. The main results and notation in vector calculus, including differential operators in curvilinear coordinate systems and an exposition of index notation, are provided in the appendices. Much of the contents of Part I will be familiar to the advanced reader but, for those who need it, it provides a preparation for most of the material to be presented in Parts II and III.

The second and third parts are more demanding and are oriented mainly toward Ph.D. students and researchers. However, advanced Masters' students and first-year graduate students in the USA will also find this material rewarding. Using explicit examples, motivated mainly by their physical interest, we explore the main themes of this book. We investigate the scaling properties of singularities as they are formed, starting from exact self-similarity and progressing to more complex forms that for example involve logarithmic corrections. We explore the structure of persistent singularities such as shocks, cusps, and vortices and finally turn to the interaction between singularities and their motion.

We wrote these two final parts of the book in the spirit of a special topics course in a postgraduate program, so that most chapters can be read independently of one another and mostly using material from the first part. We have aimed to present calculations explicitly and explain mathematical methods as we go along, but a prior knowledge of asymptotic methods such as matched asymptotic expansions and WKB methods, of basic complex variable theory, and of elements of the theory of differential equations would be helpful. Since most of the book deals with problems in continuum mechanics, some background in fluid and solid mechanics at the undergraduate level will further enhance understanding (although this is not essential since we present the main concepts and mathematical formulations in Chapter 4). We also expect the book to be useful as a "toolbox" for experienced researchers since it gathers together many ideas and techniques scattered throughout the scientific literature.

To aid self-study, we have added a number of examples to the text; these are designed to reinforce the reader's understanding of the most important concepts. The material of each chapter is supplemented by a collection of exercises of varying degree of difficulty. Some are simple extensions of material contained in the text and will be useful for self-study; others are more demanding and could be used as problems for a graduate course. Exercises which are especially demanding and explore new material have been marked with a star.

The idea of writing a unifying description of singularities was an outgrowth of the program "Singularities in mechanics: formation, propagation and microscopic description", organized with C. Josserand and L. Saint-Raymond, which took place between January and April 2008 at the Institut Henri Poincaré in

Paris. We are grateful to all participants for their input, in particular C. Bardos, M. Brenner, M. Escobedo, M. Marder, F. Merle, H. K. Moffatt, Y. Pomeau, A. Pumir, J. Rauch, S. Rica, L. Vega, T. Witten, and S. Wu. We are grateful to our editor, David Tranah of Cambridge University Press, for his encouragement to write this book and for his many suggestions along the way. We thank our colleagues and collaborators, M. Aguareles, S. Balibar, M. V. Berry, D. Bonn, M. P. Brenner, I. Cohen, S. Courrech du Pont, R. D. Deegan, L. Duchemin, T. F. Dupont, R. Evans, S. Grossmann, J. Hoppe, C. Josserand, L. P. Kadanoff, D. Leppinen, J. Li, L. Limat, J. Lister, E. Lorenceau, J. M. Martin-Garcia, G. H. McKinley, S. R. Nagel, L. M. Pismen, D. Quéré, J. H. Snoeijer, H. A. Stone, N. Suramlishvili, J. J. L. Velazquez, E. Villermaux, and C. Wagner, for their invaluable contributions toward a better understanding of singularities. The whole book was read by C. Lamstaes, who caught many errors and suggested improvements.