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# PART I

## Setting the scene

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# 1

# What are singularities all about?

Many mathematical descriptions of natural phenomena use the language of partial differential equations (PDEs). For example, fluid flow is described by the Navier–Stokes equation for the velocity, density, and pressure inside a fluid. A typical situation is shown in Fig. 1.1: a container filled with a viscous fluid is emptying through a hole in the bottom. The flow that results deforms the interface between the fluid and the air above it, whose shape is observed by lighting the container from behind and placing a camera on the other side (the interface is axisymmetric with respect to the axis of symmetry of the hole). The light passes through the fluid but is refracted by the fluid–air interface, which appears black.

The picture on the left of Fig. 1.1 conforms with the naive expectation that the scale of deformation of the interface is set by scales imprinted on the system externally, for example the size of the sink hole in the bottom (diameter $d = 1$ mm) or the minimum distance $h$ between the interface and the bottom. However, this expectation is contradicted by the picture on the right,

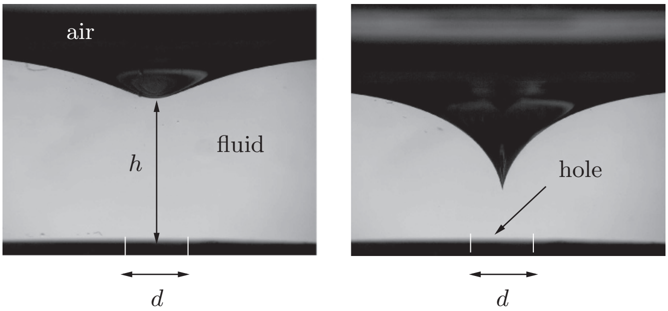


Figure 1.1 A container filled with viscous silicone oil is emptying slowly through a circular hole at the bottom [55], whose diameter $d = 1$ mm.

whose experimental conditions differ from those on the left only by the fact that the fluid has run out to a slightly lower mean level. Although the minimum distance $h$ is still comparable with the hole diameter, the free surface has deformed into a very sharp tip. The size of the tip, as measured by its radius of curvature, is below the optical resolution of about 1 μm, and extrapolation of the experimental data suggests a vanishingly small value [55].

Thus the interface is no longer smooth, and the system has reached a singularity; the curvature, which involves the second derivative of the shape, diverges as the tip is approached. All examples of singularities to be discussed in this book involve quantities diverging in either space or time (so-called *blowup*) or the divergence of some derivative of the original quantities. Intuitively, this means that a local length scale of the system goes to zero. Often this is the result of nonlinearities of the problem, which couple different length scales. As in Fig. 1.1, nonlinearities serve to *focus* the flow into very small scales, although it is driven on scales which are much larger.

Near a singularity, the characteristic length scales of the solution are ultimately smaller than the microscopic length scales of the physical system it describes, such as the size of a molecule. This calls into question the assumptions made in deriving the equation (most often a PDE) from the underlying physics. As we shall see, in many cases the solution of the differential equation still presents us with a unique and physically meaningful answer to the problem we posed originally; however, a certain (in all likelihood very small) region in space or time then needs to be excluded, for which no physical prediction can be made.

In other cases the missing microscopic information needs to be supplemented by imposing additional physical conditions in order to guarantee the *uniqueness* of the solution. Finally, there are cases in which microscopic information has to be included explicitly in order to obtain a meaningful solution.

**The fingerprint of a PDE**

As we will see throughout this book, many nonlinear PDEs exhibit a *generic mechanism* by which singularities form. Since singularities are a local phenomenon (cf. Fig. 1.1), involving arbitrarily small length scales, their structure is usually universal, i.e. independent of the initial conditions or boundary conditions imposed over macroscopic distances. In other words, singularities are the “fingerprint” of a nonlinear PDE. They are the only thing one can say about the solution that is independent of the initial or boundary conditions and thus represents the intrinsic structure of the equation itself. Let us illustrate some key features using physical examples below.

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## 1.1 Drop pinch-off: scaling and universality

Figure 1.2 shows a snapshot of the splash produced by a drop that has fallen into a pool of water, forming a nearly cylindrical column. The picture reveals that such a cylinder is unstable: even before the splash has had a chance to fall back into the pool the radius of the liquid column goes to zero at a point and a drop pinches off. Qualitatively, the reason is that the drop has a lower surface area than a piece of the cylinder occupying the same volume. To create an interface between fluid and air, one requires an energy equal to the surface tension $\gamma$ times the surface area. Thus the system can reach a lower energy state by decaying into drops.

A point of particular interest is the moment when the volume of fluid separates into two pieces, and a drop is formed. A numerical description will run into problems at that point since such a change of topology represents a discontinuous transformation. Moreover, pinch-off events determine the structure of the resulting flow; they are the crucial moments where the solution changes qualitatively. We now focus on these singularities (to be discussed in detail in Chapter 7) which occur at the time of pinch-off, which we denote by $t_0$.

### Scaling

The formation of a singularity is controlled by the time interval $\Delta t = t_0 - t$ to pinch-off. Note that we have defined $\Delta t$ to be a positive quantity for $t < t_0$, before the singularity occurs; we will come back to this point below. The dimension of surface tension is that of an energy per unit area: $[\gamma] = \mathrm{g/s^2}$. If the viscosity is sufficiently small to be irrelevant, the fluid motion is resisted only by inertia. Thus the other important parameter is the density $\rho$, representing the amount of mass being moved around, with $[\rho] = \mathrm{g/cm^3}$. Now let us assume that the singularity is characterized by a *single* length scale, which we can take as the minimum radius $h_{\min}$. Then, dimensionally, the only possible way to represent the minimum radius is [168]:

$$h_{\min} \approx A \left(\gamma \Delta t^2/\rho\right)^{1/3} \propto \Delta t^{\alpha}. \tag{1.1}$$

Thus the minimum radius behaves as a power law with scaling exponent $\alpha = 2/3$, a conclusion which has been tested experimentally with great precision; see Fig. 1.2 (right). Singularities (at which arbitrarily small length scales are produced) lack a particular length scale, so they are typically described by power laws, which are *invariant* under a change of length scale; see Section 2.4. A very simple but powerful tool to find scaling exponents is dimensional analysis, as we have just seen. A more formal exposition of this important tool is presented in Appendix C. We now illustrate it with another example.

**Example 1.1** (Nuclear explosions) In 1950, G. I. Taylor [206] calculated the propagation of an intense blast wave (caused for example by a nuclear explosion) into an ambient gas. His scaling argument was that at very high intensities the only quantity measuring the strength of the explosion is the total energy $E$ of the explosion. As far the outside temperature or pressure is concerned, they do not matter in comparison to their values inside. However, the initial density of the ambient gas atmosphere $\rho_0$ is relevant, since it measures the inertial resistance to the motion. Thus if we want to know the radius $R$ of the blast at a time $t$, the dimensions of the relevant quantities (energy, density, length, and time) are, in terms of the units of mass, length, and time:

$$[E] = \text{g cm}^2/\text{s}^2, \quad [\rho_0] = \text{g/cm}^3, \quad [R] = \text{cm}, \quad [t] = \text{s}.$$

We can eliminate the units of mass and time between $E$, $\rho_0$, and $t$ to find for $R$ a unique quantity, with dimensions of length, up to a constant prefactor $\Gamma$:

$$R(t) = \Gamma \left( \frac{Et^2}{\rho_0} \right)^{1/5}. \tag{1.2}$$

This is Taylor's result for the radius of an intense blast wave; we will calculate the prefactor $\Gamma$ in Exercise 11.11.

Repeating the same argument using the language of the Buckingham $\Pi$-theorem (see Appendix C), we have $k = 4$ quantities and $m = 3$ independent units. Since $k - m = 1$, we can construct a single dimensionless quantity (called a dimensionless group)

$$\Pi = \frac{R\rho_0^{1/5}}{E^{1/5}t^{2/5}}.$$

According to (C.2) we have $\phi(\Pi) = 0$ for a suitably defined function $\phi$. But this is consistent only if $\phi(\Pi) \equiv \Pi - \Gamma = 0$ for some constant $\Gamma$, which once more implies (1.2). □

Before we go on, let us point to the considerable subtlety that often underlies dimensional arguments. Frequently, assumptions are made on physical grounds which in general have to be confirmed by a detailed analysis. In the case of drop breakup, we have assumed that there exists a *single* length scale $h_{min}$ characterizing the breakup. Dimensional analysis then demonstrates that the scaling exponent $\alpha$ is determined uniquely by the local structure of the equations alone. This scenario is known as *self-similarity of the first kind* [14].

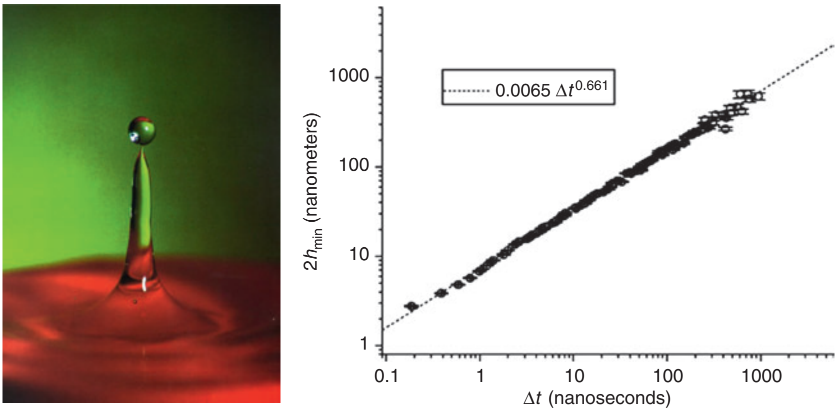


Figure 1.2 On the left, a splash of water at the moment of pinch-off. Photograph by Harold Edgerton. Copyright 2010 MIT. Courtesy of MIT Museum. On the right, the minimum diameter of a mercury drop as a function of the time interval from pinch-off. Reprinted with permission from [38]. Copyright 2004 by the American Physical Society.

However, this need not be the case, and the solution for inviscid breakup may be governed by two or more intrinsic length scales. This happens in the breakup of a two-dimensional liquid *sheet*, in which case the minimum sheet thickness and the typical width of the pinch region scale differently [39]. The ratio of these two length scales is a dimensionless number and thus cannot be fixed by dimensional analysis. As a result, the thickness and width of the sheet may shrink according to different scaling exponents, to be determined as part of the full solution to the problem. This case is known as *self-similarity of the second kind*. In general, the exponents no longer assume rational values but will become irrational numbers.

In reference to Example 1.1, the superficially very similar problem of the *focusing* of a shock wave onto a point is an example of self-similarity of the second kind [133]: the radius of the shock wave is governed by an irrational exponent, different from that in (1.2). We will return to this problem in Chapter 11 on shock waves.

**Universality**

Another, related, property of a singularity is the fact that its structure is insensitive to initial conditions or other aspects of the large-scale structure of the solution. This is a consequence of the fact that the singularity arises from a local balance: in drop pinch-off, the dimensionless prefactor $A$ in (1.1) is expected to be universal since (1.1) is the solution of a nonlinear equation.

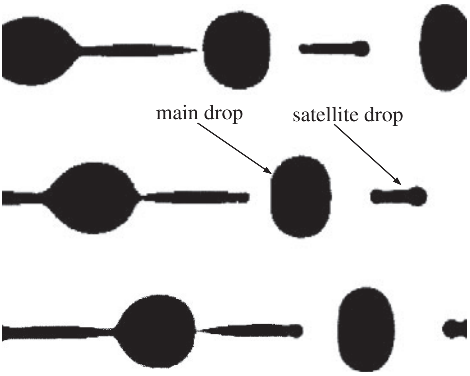


Figure 1.3 Satellite formation in a water–glycerol jet, showing a satellite drop in between two main drops. A satellite drop is the remnant of the elongated neck between two main drops [75, 146].

Any change in amplitude would disturb the balance between different terms in the equation. A detailed solution of the equation of motion shows that $A$ has a numerical value close to 0.7; see (7.98) below.

However, the concept of universality goes much deeper and governs the entire spatial structure of a singularity. This is illustrated nicely by the breakup of a liquid jet, shown in Fig. 1.3. It is evident that there are not only main drops produced in the process, but also much smaller, so-called "satellite" drops. Such a drop is formed from an elongated neck between two adjacent main drops. The existence of the neck, in turn, is related to the fact that the profile of the jet near the point of breakup is extremely asymmetric: toward the drop it is very steep while toward the neck it is flat, forcing the neck into its slender shape.

Thus the existence of satellite drops is a consequence of the nonlinear properties of the fluid motion close to breakup, which produces the asymmetric shapes discussed above.

Drop formation of the type illustrated in Fig. 1.3 has many applications, as reviewed in for example [16]. The classical technique of ink-jet printing has been adapted to produce so-called microarrays for DNA analysis, to print integrated circuits, and to produce miniature lenses for optical applications. For all these applications it is very important to control the size of the drop accurately; thus satellite drops are detrimental to the quality since they result in at least two *different* droplet sizes. It is therefore natural to ask whether it is possible to control the excitation of the jet leading to breakup in such a way that only one type of drop is produced.

A hypothetical, more desirable, breakup configuration is illustrated schematically in Fig. 1.4, in which the profile is assumed symmetric with respect to the

Figure 1.4 A hypothetical breakup mode without satellite formation: an engineering dream that is unfulfillable.

pinch point so that breakup would occur in the middle, between two drops. In that case the neck would snap back toward each of the drops, which would receive the same amount of mass, making all drops equal. However, universality imposes that such a scenario is impossible: no matter how the jet is excited or what the initial condition may be, the pinch-off dynamics will always be similar to Fig. 1.3, *independently* of the initial conditions.

**Continuation**

To fully understand drop formation as shown in Fig. 1.3, one also needs to address the problem of *continuation* across the singularity, treated in Chapter 10. As a drop is formed, one proceeds from a simply connected domain (before pinch-off) to a multiply connected domain (after pinch-off), which is a discontinuous process. As a result, it is not clear whether this continuation is unique, i.e. that there is a single post-breakup solution, with which the pre-breakup solution can be continued. We will show that in the case of drop formation there is indeed only one unique continuation which does not involve discontinuities at a finite distance away from the singularity.

## 1.2 Stationary cusps: persistent singularities

The pinch-off singularity of a drop dominates the flow for a brief period of time, as the piece of fluid breaks into two. We now discuss another type of singularity, which is stationary and which is a generic feature for a range of parameters. As an example, consider the intensity of light generated by the reflection from a wedding ring, shown in Fig. 1.5 (left). The intensity is far from uniform but instead becomes very large along certain line-like singularities called caustics (from the Latin for "burning"). The caustic line itself has a singular tip, where it ends in a cusp. We will investigate the nature of the cusp in Chapter 14, where we show that its shape is described by the universal power law

$$y \propto x^{2/3}, \tag{1.3}$$

where $x$ is the width of the cusp and $y$ the distance from its apex.

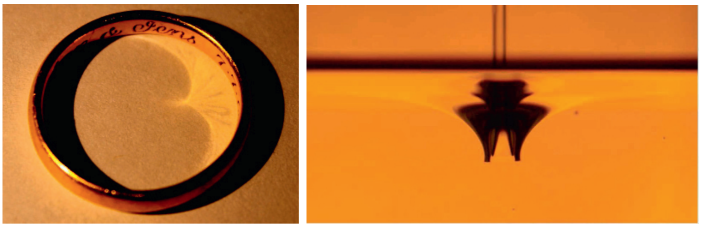

Figure 1.5 Two examples of cusp singularities, observed in very different contexts. On the left, the bright lines of a caustic produced by the focusing of wave fronts (image by Ann Eggers). On the right, a jet of viscous fluid is poured into a bath and imaged from the side, looking up toward the surface of the bath. (Reprinted with permission from [184]. Copyright 2008, AIP Publishing LLC.) The free surface ends in a cusp singularity, which lies on a circle with the jet axis at its center.

A very different problem, also treated in Chapter 14, is shown in Fig. 1.5 (right). A jet of viscous fluid falls into a bath of the same fluid, creating a strong flow. The free surface deforms into a cusp, which ends in a circular knife edge at the bottom. An analysis of this problem, which requires the solution of the viscous flow equations with a free surface, yields exactly the same cusp shape, (1.3), as for the coffee cup caustic. Thus the concept of universality may in some cases apply more broadly, connecting phenomena of very different physical origin.

One important aspect in which the two problems shown in Fig. 1.5 differ is the mechanism by which the singularity is cut off on a small scale. The motion of wave fronts, which results in the caustic, is described by a *nonlinear* equation, which *emerges* from the linear wave equation on a scale greater than the wavelength of light [21, 22]. As one approaches the caustic the light intensity, instead of diverging, dissolves into a characteristic diffraction pattern, which we will study in Section 14.5. However, the singularity at the tip of the free surface cusp is resolved by the smoothing effect of the surface tension $\gamma$. Nevertheless $\gamma$ does not by itself introduce a particular length scale. The surprising result of a more detailed calculation reveals that the radius of curvature of the cusp is in fact *exponentially small* in the value of the surface tension, i.e. it is proportional to $\exp(-\gamma^{-1})$ [122]!

## 1.3 Shock waves: propagation

If a singularity persists for a finite period of time, the question arises how it will move in space. Among the examples considered in the third part of the book