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1. Astrophysical Magnetic Fields: Essentials

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Abstract

Here we provide a general introduction to the physical aspects of astrophysical magnetic fields, presenting the widely used MHD approximation, and discussing concepts like advection, diffusion, and the operation of magnetic dynamos. During the Winter School (WS), the introductory lessons were delivered by Professor F. Cattaneo. Rather freely, these introductory notes roughly follow the thread of arguments he presented at the WS. Our *essentials* contain a number of references to classic books on astrophysical magnetic fields, so that the interested reader can expand the cursory description given here.

1.1 Astrophysical Plasmas

Only 5% of the energy density of the Universe is provided by baryonic matter (e.g., Fukugita & Peebles, 2004); however, it is the *luminous matter* and so it is our interface to study the Universe. Most of it is in the form of a plasma, where atoms and atom aggregates are partly ionized. Particles move under the influence of large-scale electromagnetic fields created collectively. The presence of magnetic fields in plasmas is pervasive, and one cannot be properly understood without the other. This holds from planets to the Universe as a whole, including stars and the interstellar medium, galaxies, clusters of galaxies, and the intergalactic medium. The conditions vary from object to object, and the resulting phenomena depend on the temporal and spatial scales of interest. Fortunately, a few equations capture (albeit in a simplified way) many of the physical processes that are characteristic of magnetized plasmas. They will be introduced here in preparation for the specific applications to be described in the forthcoming chapters. The main equations will be given without any proof or derivation; these can be found in many of the monographs on astrophysical plasmas and magnetohydrodynamics (MHD) existing in the literature, e.g., Parker (1979), Moffat (1978), Priest (2000), Cattaneo (1999), Kulsrud (2005), or Spruit (2013). The symbols used in this description are summarized later in the chapter, in Table 1.2.

• *How can you describe the plasma mathematically?* Plasmas are made out of neutrals, electrons, ions, and electromagnetic fields. By and large, a plasma can be characterized by the number density of each species n_i, the temperature of each species T_i, and the steady-state magnetic field $\vec{B}$. At equilibrium, the temperature must be the same for all species; however, the relaxation time to return to equilibrium may be long. Often one will have high energetic processes, localized in time and space (e.g., reconnection in a solar flare, or ejection of plasma blobs in magnetized jets), that move the system out of thermodynamic equilibrium. In returning to equilibrium, each species is often in equilibrium with its own temperature T_i, allowing us to assume a Maxwellian velocity distribution for each species separately. Nature provides situations where equilibrium is never reached, e.g., in turbulence.

1.1.1 *Characteristic Scales of Plasmas*

The physical properties and behavior of a plasma depend on a number of characteristic spatial and temporal scales. Their values in a number of representative astrophysical contexts are given in Table 1.1.

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TABLE 1.1. Physical parameters characteristic of laboratory and astrophysic

Physical parameter	Magnetic fusion	Accretion disk (Sgr A)	Solar chromosphere	Interstellar medium
Electron density [cm^{-3}]	10^{14}	10^{6}	2×10^{10}	1
Gas temperature [K]	10^{8}	10^{11}	10^{4}	10^{4}
Magnetic field strength [G]	5×10^{4}	50	10	10^{-6}
Plasma parameter	10^{8}	5×10^{16}	10^{4}	10^{9}
Plasma frequency [Hz]	5×10^{11}	5×10^{7}	10^{10}	5×10^{4}
Mean-free path [cm]	10^{6}	10^{20}	50	10^{12}
Gyroradius electrons [cm]	5×10^{-3}	10^{2}	10^{-1}	2×10^{6}
Plasma beta	10^{-2}	10^{-1}	10^{-2}	50
Characteristic spatial scale [cm]	10^{2}	10^{13}	10^{8}	10^{20}

See, e.g., Schekochihin *et al.* (2009) or Alonso de Pablo & Sánchez Almeida (2013).

- *Debye length.* Most plasmas are quasi-neutral, i.e., they have the same number of free electrons as the number of positive charges in ions. However, this quasi-neutrality goes away when you average over the very small scales able to single out individual electrons or ions. The Debye length is the minimum scale at which the plasma looks neutral. A free charge in a neutral plasma creates an electric potential at a distance r from the charge $\phi(r)$, which is the electric potential in a vacuum $\phi_0(r)\big(\propto r^{-1}\big)$ plus an exponential drop-off,

$$\phi(r) = \phi_0(r)\, \exp(-r/\lambda_D), \tag{1.1}$$

so that when $r > \lambda_D$ then $\phi \ll \phi_0$, and the effect of the point charge becomes negligible shielded by the plasma. The characteristic scale for shielding the charge is the Debye length,[1]

$$\lambda_D = \sqrt{\frac{kT_e}{4\pi\, n_e e^2}}. \tag{1.2}$$

In order to treat a plasma as quasi-neutral, one has to assume that the characteristic length of interest is much larger than λ_D.

- *Plasma frequency.* If, for any reason, ions and electrons are separated, their separation oscillates to try to recover the quasi-neutrality with a frequency called the plasma frequency, w_e

$$w_e = \sqrt{\frac{4\pi\, n_e\, e^2}{m_e}}. \tag{1.3}$$

The period associated with this frequency sets the timescale at which the plasma responds to quasi-neutrality, so that deviations from neutrality last less than this timescale. Note that the frequency given in Eq. (1.3) corresponds to the electron plasma frequency. Ions also respond to deviations from neutrality, but they oscillate at a much lower frequency ($w_p^2 \propto m_p^{-1}$), therefore, electrons set the overall characteristic timescale.

- *Plasma parameter.* The number of electrons in a volume with the radius of the Debye length is the plasma parameter Λ,

$$\Lambda = \frac{4\pi}{3}\, \lambda_D^3\, n_e = \frac{1}{3\sqrt{4\pi\, n_e}} \left(\frac{kT_e}{e^2}\right)^{3/2}. \tag{1.4}$$

$\Lambda \gg 1$ in astrophysical plasmas (see Table 1.1), i.e., there are many free electrons per Debye sphere. Each individual electron feels the Coulomb forces of many electrons simultaneously. Electrons do not interact strongly with each other, but feel the global potential created collectively by the plasma. Each electron random-walks with such small deflections that only a large number of them will produce significant changes in the original trajectory. The plasma is said to be weakly coupled. One can also show that the electrostatic potential energy is much smaller than the kinetic energy of the electrons (and the rest of the species). $\Lambda \gg 1$ implies that the plasma is "hot" and "tenuous" (high temperature and low density: see, e.g., Eq. (1.4)).

- *Gyro frequency and gyro radius.* An electron moving in a magnetic field follows a helical path around the axis set by the magnetic field vector. The gyro frequency of

[1] In this definition and throughout the chapter, equations are given in cgs-esu units following, e.g., Somov (2006).

this motion, Ω, is

$$\Omega = \frac{e\,B}{m_e\,c}, \tag{1.5}$$

with the gyro radius (or Larmor radius) given by

$$r_L = \frac{v_\perp m_e\,c}{e\,B}, \tag{1.6}$$

where $v_\perp$ is the component of the electron velocity perpendicular to the magnetic field. For the typical electron thermal energies and magnetic fields found in astronomy, the gyro radii are much smaller than the characteristic scale for the variation of the magnetic fields (see Table 1.1).

- *Mean free path.* The description of MHD plasmas requires having enough collisions between electrons, ions, and neutrals to grant the coupling of their properties. For this condition to be granted, the distance travelled between collisions must be much smaller than the characteristic scales for the variation of the macroscopic physical parameters such as magnetic field, temperature, or bulk velocity. The mean free path between collisions λ is

$$\lambda = \frac{1}{n_e\sigma}, \tag{1.7}$$

where σ is the Coulomb cross section, which is similar for electrons and ions (e.g., Kulsrud, 2005),

$$\sigma \simeq 10^{-12}\,\mathrm{cm}^2\,T_{\mathrm{eV}}^{-2}, \tag{1.8}$$

where T_{eV} stands for the temperature in eV ($1\,\mathrm{eV} \equiv 10^4\,\mathrm{K}$). The mean free paths in astrophysical plasmas are generally larger than the lengthscales characterizing the variation of macroscopic parameters; see Table 1.1.

- *Plasma β.* This is the ratio between the gas pressure P and the magnetic pressure,

$$\beta = \frac{8\pi P}{B^2}, \tag{1.9}$$

and it parameterizes whether magnetic forces ($\beta \ll 1$) or hydro forces ($\beta \gg 1$) dominate the behavior of the plasma.

Table 1.1 contains the typical physical parameters of various astrophysical plasmas. As a consequence of the large range of parameters, the physical behavior of the astrophysical plasma is very diverse. Before starting work on a particular subject, one has to work out the characteristic plasma scales, to be sure that the (approximate) equations to be used are appropriate for the problem at hand. For example, the atmospheres of collapsed objects such as white dwarfs and neutron stars do not have $\Lambda \gg 1$.

1.1.2 *Mathematical Description of Plasmas – MHD – Induction Equation*

The exact solution for the motion of each particle under the influence of local magnetic fields is extremely difficult to obtain. In practice, one uses approximate equations that involve averages of the exact solution. Depending on how such an averaging is carried out, one gets different approximations. Under the *Vlasov theory*, one averages over all particles of a given species with the same velocity at a given location. The properties of the plasma are then described in terms of distribution functions. In *multifluids theory* one averages over all particles of a given species for all velocities. One gets the mean densities and velocities for each species. Temperature and pressure emerge naturally. Finally, the *magnetohydrodynamic (MHD) approximation* is based on averages over all particles of all species and velocities. The properties of the plasma are then described in terms of mean density, velocity, and pressure at each spatial location.

Within MHD, one finds the equations for mass conservation, momentum conservation, and energy conservation typical of the fluid dynamics. Electro-magnetic fields are considered, starting from the Maxwell equations under the assumptions of (1) quasi-neutrality, (2) velocities that are small with respect to the speed of light, and (3) currents proportional to the electric field in the rest-frame of the current-carriers (see the references to textbooks given in the previous section). This approximation holds for plasmas where the different species are coupled by collisions, thus giving the character of a fluid to the plasma. With the above simplifications, one derives the *induction equation*, which is the fundamental equation in MHD,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \wedge (\vec{v} \wedge \vec{B}) + \eta \nabla^2 \vec{B}, \tag{1.10}$$

where η is the magnetic diffusivity. This is the inverse of the electric (Ohmic) conductivity, and has units of a velocity times a length (it is a diffusion coefficient). The symbols $\vec{B}$ and $\vec{v}$ stand for the magnetic field and the bulk velocity of the plasma, respectively. This shows how the magnetic field evolves with time t given the velocity field and the diffusivity. The first term on the right-hand side of the induction equation creates magnetic fields from plasma motions, and contains the inductive and advective effects of the velocity on the magnetic field. The second term is a diffusive term that dissipates magnetic energy. The ratio between the two of them is the magnetic Reynolds number R_m,

$$\frac{|\nabla \wedge (\vec{v} \wedge \vec{B})|}{|\eta \nabla^2 \vec{B}|} \sim \frac{v\, B/\mathcal{L}}{\eta B/\mathcal{L}^2} \sim \frac{\mathcal{L} v}{\eta} = R_m, \tag{1.11}$$

where $v = |\vec{v}|$ and $\mathcal{L}$ is a characteristic lengthscale of the problem. R_m controls the relative importance of the advective and the diffusive processes. The properties of the magnetic fields satisfying the induction equation are analyzed in the next section.

1.1.3 *General Properties of the Induction Equation*

In many common astrophysical contexts the magnetic Reynolds number is very large $R_m \gg 1$, meaning that the last term in the right-hand side of Eq. (1.10) is negligible and the physics is advection dominated. Often this case is referred to as *ideal* MHD. In this case a number of interesting properties arise; in particular, the magnetic field lines are *frozen* in the plasma. Magnetic field lines are material lines, and they move by being dragged along with the fluid motion. The magnetic flux through any closed co-moving surface is constant.

- *Consequences of magnetic flux conservation.* Suppose that a star collapses to become a white dwarf conserving the magnetic flux, i.e., conserving $B\,\mathcal{L}^2$, where $\mathcal{L}$ is the size of the object. If a solar-size star ($\mathcal{L} \sim 10^{11}$ cm) collapses to become white-dwarf sized ($\mathcal{L} \sim 10^9$ cm), the magnetic field increases from, say, 10 G to 10^5 G. If the collapse goes all the way down to a neutron-star size object ($\mathcal{L} \sim 10^6$ cm), then the magnetic field booms up to 10^{11} G. These huge field strengths are indeed observed (e.g., Mereghetti, 2008).
- *Consequences of the flux freezing on the magnetic helicity.* These consequences are not as intuitive as for the magnetic flux conservation. The magnetic helicity $\mathcal{H}$ is defined as

$$\mathcal{H} = \int_u \vec{A} \cdot \vec{B}\, dV, \tag{1.12}$$

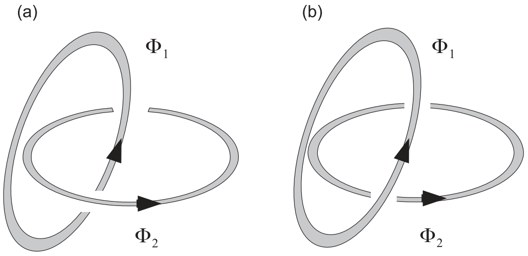

FIGURE 1.1. Schematic to illustrate the physical meaning of helicity and its conservation under frozen-in conditions. (a) The helicity of these two nested magnetic rings is $\mathcal{H} = 2\Phi_1\Phi_2 \neq 0$. The symbols Φ_1 and Φ_2 stand for the magnetic flux across the cross section of the rings 1 and 2, respectively. (b) In this case $\mathcal{H} = 0$ – the magnetic loops are similar to the previous ones but this time they are not intertwined.

with $\vec{A}$ the vector potential ($\vec{B} = \nabla \wedge \vec{A}$). The volume integral has to be bounded by a surface having $\vec{B}$ always parallel to the surface. (If $\vec{n}$ is the normal to the surface, then $\vec{B} \cdot \vec{n} = 0$.) Under the frozen-in condition, the helicity is conserved, so that

$$\frac{d\mathcal{H}}{dt} = 0. \tag{1.13}$$

How do we interpret this? The frozen-in condition does not allow the fluid evolution to modify the magnetic field topology. The situation is illustrated in Fig. 1.1. It can be proven that the helicity in Fig. 1.1a is the product of the fluxes in the two knotted magnetic fluxtubes (rings), i.e., $\mathcal{H} = 2\Phi_1\Phi_2$ (e.g., Moffat, 1978). In the case of Fig. 1.1b, $\mathcal{H} = 0$, implying that when the magnetic field is frozen in, it is imposible to change from the topology in Fig. 1.1a to topology in Fig. 1.1b. Diffusion is needed for the magnetic field lines to cross each other, and so for the helicity to change.

- *The effect of diffusion.* Without diffusion, the magnetic field would keep the topology from the moment when it was created at the origin of the Universe. Therefore, even if throughout space $R_m \gg 1$, there should be regions where $R_m \leq 1$, where the magnetic field is diffusive. These regions are critical to allow a change in magnetic field topology, and to allow creation and destruction of $\vec{B}$. (For example, to explain the magnetic cycles existing in stars like the Sun.) In the absence of velocity fields, the timescale for the magnetic field to diffuse τ is approximately given by

$$\tau \sim \frac{B}{|\partial\vec{B}/\partial t|} \sim \mathcal{L}^2/\eta, \tag{1.14}$$

where we have used the induction Eq. (1.10) without the advection term, with $|\nabla^2\vec{B}| \sim B/\mathcal{L}^2$. These timescales are huge in astronomical contexts, often larger than the age of the Universe, because the lengthscales $\mathcal{L}$ that are involved are astronomically large. Therefore this diffusion timescale is physically irrelevant. However, when velocity fields are considered, they are able to generate tiny lengthscales on the magnetic field – so small that diffusion occurs on timescales of the order of the dynamical timescale of the system, and become physically relevant. What is the lengthscale ζ at which diffusion balances advection? Setting $\partial\vec{B}/\partial t = 0$ in the induction equation,

$$\zeta \sim \mathcal{L}/R_m^{1/2}, \tag{1.15}$$

which implies that, for the typical $R_m \gg 1$, ζ is tiny. Therefore, the changes in the magnetic field topology have to occur at these tiny lengthscales for the diffusion to be effective. But the existence of these scales completely changes the timescale of the diffusion process, so that the timescale τ_ζ is no longer τ (Eq. (1.14)) but it turns out to be much shorter, as given by

$$\tau_\zeta \sim \zeta^2/\eta \sim \tau\, R_m^{-1}. \tag{1.16}$$

- *Turbulent diffusion.* When the velocity field has a turbulent component of amplitude v_t that varies over lengthscales l_t, the plasma behaves as if it has an effective diffusion called *turbulent diffusion*, given by

$$\eta_t \simeq l_t v_t. \tag{1.17}$$

  Often $\eta_t \gg \eta$, so that the turbulent diffusion dominates the diffusion processes in astrophysical plasmas.
- *The two competing roles of the plasma velocity.* The velocity field both speeds up the magnetic diffusion processes, and increases the magnetic field. The increase of magnetic field gradient leads to accelerated decay, and the stretching of field-lines leads to magnetic field growth. Both stretching and creation of gradients have exponential growth in time. Which one wins depends on the flow. If stretching wins, then we have a dynamo. When do we have it? There is no general rule and the answer has to be worked out on a case-by-case basis; however, any complicated velocity field at high R_m produces a dynamo (e.g., Childress & Gilbert, 1995). This is the context of many astrophysical plasmas.

1.2 Astrophysical Dynamos

If a magnetized plasma has no motions, then the magnetic field will decay with a timescale given by either Eq. (1.14) or Eq. (1.16). Fluid motions may increase these timescales to make them infinitely large. A velocity field $\vec{v}$ has *the dynamo property* if the energy of the magnetic field of the plasma does not decay with time, i.e., if

$$\int_{\text{Volume}} B^2\, dV \rightarrow \text{constant} \neq 0 \text{ for } \text{time} \rightarrow \infty. \tag{1.18}$$

Dynamos are supposed to be responsible for most astrophysical magnetic fields, from planets to the early Universe. Are dynamos universal objects? Is the mechanism that makes a dynamo work always the same? Unfortunately, dynamos are not universal and different mechanisms may operate to amplify magnetic fields. (Even though all complicated velocity fields tend to have the dynamo property.)

An important distinction to be made is the difference between large-scale and small-scale dynamos. If the typical lengthscale of the velocity is l_v, and the lengthscale of the generated magnetic field is l_B, the dynamo is small-scale if $l_B \ll l_v$. Otherwise, the dynamo is said to be large-scale. Take the Sun as an example. Understanding the dipolar component of the solar magnetic field involves a large-scale dynamo since there are no plasma motions with the scale of the full Sun (except for rotation). On the other hand, the small-scale magnetic field observed in the Sun (less than 100 km across) may be created by a small-scale dynamo if it is generated by solar granulation (1000 km size). Large-scale dynamos generate ordered magnetic fields, whereas small-scale dynamos produce tangled magnetic fields. For this reason, large-scale dynamos are said to generate magnetic flux, whereas small-scale dynamos generate magnetic energy.

- *Mean-field dynamo.* The mean-field theory, or mean-field electrodynamics, is described in many classical text books, e.g., Moffat (1978) or Krause & Rådler (1980). It is based on a two-scale approach. If this two-scale approximation fails, then the whole theory collapses. The two-scale approach is illustrated in Fig. 1.2. The magnetic field and

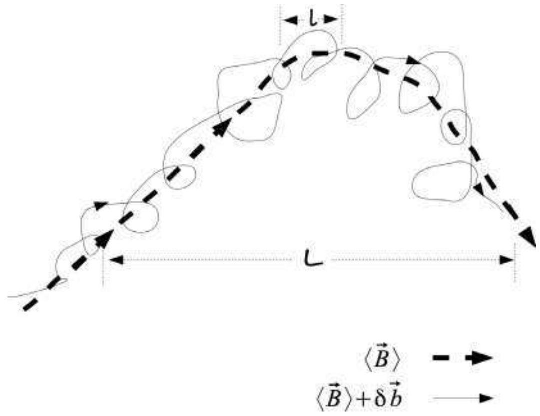

FIGURE 1.2. Schematic illustrating the two-scale approach to the mean-field dynamo theory. The magnetic field is divided into a mean field ($\langle\vec{B}\rangle$; the thick dashed line) and a fluctuating part ($\delta\vec{b}$), so that the total magnetic field (the thin solid line) is the sum of these two parts. The lengthscale for variation of the mean field (L) is much larger than the scale for the fluctuating part ($l \ll L$).

the velocity can be divided into a large-scale mean quantity and a fluctuating quantity,

$$\vec{v} = \langle\vec{v}\rangle + \delta\vec{v}, \tag{1.19}$$

$$\vec{B} = \langle\vec{B}\rangle + \delta\vec{b}, \tag{1.20}$$

with the angle brakets $\langle\ \rangle$ meaning averaging over a scale a in between the characteristic scale for the variation of the average quantities L, and the characteristic scale for the variation of the fluctuations l ($\ll a \ll L$). There is a gap in physical scales, and the average is carried out with a smearing scale between them. By definition, the fluctuations average out to zero,

$$\langle\delta\vec{v}\rangle = \langle\delta\vec{b}\rangle = 0. \tag{1.21}$$

Then the induction equation (Eq. (1.10)) yields,

$$\frac{\partial\langle\vec{B}\rangle}{\partial t} = \nabla\wedge\left[\langle\vec{B}\rangle\wedge\langle\vec{v}\rangle + \langle\delta\vec{v}\wedge\delta\vec{b}\rangle\right] + \eta\,\nabla^2\langle\vec{B}\rangle, \tag{1.22}$$

which is formally identical to the original Eq. (1.10) considering average quantities, plus an extra advective term created by the correlation of the fluctuating parts of the fields,

$$\epsilon = \langle\delta\vec{v}\wedge\delta\vec{b}\rangle. \tag{1.23}$$

In order to close the system of equations provided by the average induction equation (Eq. (1.22)), one needs to express ϵ in terms of the average magnetic field. The symbol ϵ is an electromotive force. It is possible to show that there should be a linear relationship between the electromotive force and the mean magnetic field. The most general expression is given by

$$\epsilon_i = \alpha_{ij}\langle B_j\rangle + \beta_{ijk}\frac{\partial}{\partial X_j}\langle B_k\rangle + \cdots \tag{1.24}$$

Since the average quantities are slowing varying functions of space coordinates X_j, the higher-order terms can be neglected. (This approximation is valid as long as the scale separation holds.) The tensors α_{ij} and β_{ijk} depend only on the velocity of the plasma and the conductivity, but not on $\langle B_j \rangle$. Assuming the velocity is turbulent but homogeneous (i.e., does not depend on position) and isotropic (i.e., does not depend on direction), then

$$\alpha_{ij} = \alpha\, \delta_{ij}, \qquad \beta_{ijk} = -\beta\, \epsilon_{ijk}, \tag{1.25}$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the completely antisymmetric rank-three tensor. If these expressions are plugged into the average induction equation (Eq. (1.22)), with $\langle \vec{u} \rangle = 0$ and with α and β constant for simplicity, one ends up with

$$\frac{\partial \langle \vec{B} \rangle}{\partial t} = \alpha\, \nabla \wedge \langle \vec{B} \rangle + (\eta + \beta)\, \nabla^2 \langle \vec{B} \rangle, \tag{1.26}$$

where $\nabla \wedge \langle \vec{B} \rangle$ is the mean current. So, there is a turbulent diffusion term (β) plus a source of magnetic field aligned to the main current (α term). If $\alpha \neq 0$, Eq. (1.26) tells us that dynamos are to be expected with the following order-of-magnitude argument: the diffusive terms drops as the square of the lengthscale L for the variation of the mean field ($|\nabla^2 \langle \vec{B} \rangle|$ is of the order of B/L^2), whereas the positive α-term drops as L ($|\nabla \wedge \langle \vec{B} \rangle|$ goes as B/L). Therefore, for $L \to \infty$, the source term wins, $\partial \langle \vec{B} \rangle / \partial t > 0$, and one has a dynamo.

How do we know that $\alpha \neq 0$? The parameter α must change sign under parity transformation as $\vec{B}$ does. This is just a necessary condition, but unfortunately no sufficient condition for $\alpha \neq 0$ is known to exist. Purely turbulent velocity fields do not fulfill the requirement since they lack reflexion symmetry. However, turbulence in rotating bodies does fulfill it, and thus rotation is an important ingredient in many astronomical dynamos. An intuitive account of the α effect in the Sun was put forward in a fundamental paper by Parker (1955). A velocity field is able to create currents parallel to the mean magnetic field, thus producing non-zero α. The mechanism is sketched in Fig. 1.3. The magnetic fields are stored at the bottom of the convection zone, and they can be dragged along by the convective plumes that continuously rise from the bottom toward the solar surface. Buoyancy plus Coriolis forces move up and rotate the plasma traveling through the solar convection zone, like the cyclonic events shown in Fig. 1.3. These cyclonic motions lift and twist the magnetic field lines as shown in Fig. 1.3a, creating a curled magnetic field with a current antiparallel to the mean field. Stronger cyclonic events produce currents that are parallel to the mean field (Fig. 1.3b). The collective effect of many cyclonic events produces the α effect.

For any velocity field to produce an α effect, there should be a non-trivial correlation between the velocity to go up and around, i.e., a non-trivial correlation between the velocity and its vorticity. In technical terms, its kinetic helicity K has to be different from zero, i.e.,

$$K = \langle \delta\vec{v} \cdot \nabla \wedge \delta\vec{v} \rangle \neq 0. \tag{1.27}$$

Given a velocity field, it is non-trivial to infer the value of α since this is equivalent to solving the induction equation for the magnetic field perturbations. It can be done in cases with, e.g., small magnetic Reynolds number, which are not relevant in the astrophysical context.

The mean-field dynamo may present a problem in astrophysical contexts, where R_m is large. The linear analysis of the growth rates of the various dynamo modes shows that the mode of maximum growth rate is the one that has the lengthscale of the fluctuating velocity field. This uncovers an internal inconsistency of the theory because the mean-field electrodynamics implies a two-scale approach that, ultimately, allows us to neglect all the high-order derivatives in the expansion (1.24). The importance of this problem is not clear yet. The mean-field dynamo works for diffusive plasmas (small R_m).

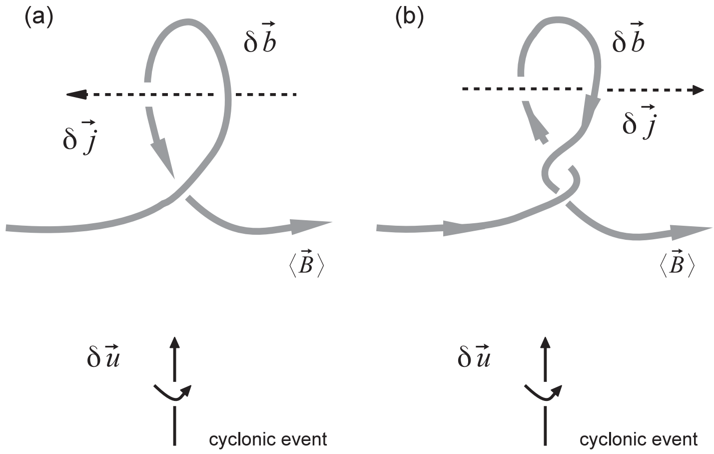

FIGURE 1.3. Parker's explanation for the α effect on the Sun. (a) Individual *cyclonic events* (velocity perturbation depicted at the bottom) move up and twist the magnetic field lines. It creates a fluctuation in the magnetic field $\delta\vec{b}$ with an associated current $\delta\vec{j}$, which is antiparallel to the mean field $\langle\vec{B}\rangle$, i.e., as required for the mean-field dynamo to operate. (b) If the cyclonic event is stronger, then the currents can become parallel to the mean field.

- *Small-scale dynamo.* The arguments mentioned above refer to large-scale magnetic fields, where their scale is larger than the characteristic scale of the velocity field producing the dynamo. What about small-scale fields? How do we generate magnetic energy, even in flows that cannot generate large scale magnetic fields? This is the small-scale dynamo, also extremely important in astrophysical contexts.

In an incompressible random plasma of large R_m, the magnetic field becomes locally transformed as it moves along with the flow as dictated by the Jacobian. If the flow is chaotic, the Jacobian corresponding to each parcel of the fluid is completely different so that, in the end, the magnetic field is modified according to the product of a large number of random matrices. One can prove that this process corresponds to an exponential expansion or stretching of the magnetic field in three orthogonal directions. The rate of expansion-stretching is given by the three Liapunov exponents

$$\lambda_1 > \lambda_2 > \lambda_3, \tag{1.28}$$

with the condition

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{1.29}$$

needed to preserve the volume. If the magnetic field is stretched in one direction, it has to be compressed in other(s). One can think of λ_1 as the increase of length (per unit length) and of $|\lambda_3|$ as the increase of the gradients. The stretching increases the field strength whereas the gradients augment the diffusion, so that there will be a dynamo if the stretching wins, and this seems to happen very often when the velocity field is complex enough (e.g., Cattaneo, 1999), provided the magnetic Reynolds number is large.

The small-scale dynamo operates even when the viscosity is very small. The magnetic Prandtl number P_m, i.e., the ratio between the viscosity of the flow ν and the magnetic diffusivity η, is basically a measurement of the ratio between the lengthscale of $\vec{B}$, l_B, set by magnetic diffusion, and the lengthscale of $\vec{v}$, l_v, set by viscosity,

$$P_m = \frac{\nu}{\eta} = (l_v/l_B)^2, \tag{1.30}$$

where this expression considers that the timescales for changing $\vec{B}$ and $\vec{v}$ have to be similar (see Eq. (1.14); and keep in mind that the timescale for changing the velocities is formally similar to τ replacing η with ν). Consequently,

$$P_m \begin{cases} \ll 1 & \text{if random } \vec{v} \text{ at the characteristic scale of } \vec{B}, \\ \gg 1 & \text{if smooth } \vec{v} \text{ at the characteristic scale of } \vec{B}. \end{cases} \tag{1.31}$$

In the Universe, P_m is either very small (e.g., stellar interiors) or very large (e.g., interstellar medium). Only in simulations does P_m tend to be around one. Ten years ago it was found that, when P_m was significantly smaller than one, the turbulent dynamo in the numerical simulations switched off (e.g., Brandenburg, 2011, and references therein). For some time it was believed that small P_m dynamos were unfeasible, but this seems to be an artifact due to technical limitations. The computational box needed to capture the dynamo increases with the roughness of the velocity field. If you decrease P_m, everything else being constant, the roughness increases to a point that the computational domain is insufficient to maintain the dynamos, and the dynamo in the numerical simulations dies out. It can be proven using analytical solutions of the induction equation that dynamo action is possible at any P_m if R_m is large enough.

1.3 Symbols used in the chapter

TABLE 1.2. List of the main symbols used in the chapter

Symbol	Meaning
a	lengthscale for the averages in mean-field theory
$\vec{A}$	vector potential, i.e., $\nabla \wedge \vec{B}$
$\vec{B}$	magnetic field vector
B	magnetic field strength, i.e., $\lvert\vec{B}\rvert$
B_i	ith component of $\vec{B}$
β	plasma beta
$\delta\vec{b}$	fluctuating magnetic field in mean-field dynamo
$\delta\vec{u}$	fluctuating velocity field in mean-field dynamo
$\delta\vec{j}$	fluctuating current in mean-field dynamo
δ_{ij}	Kronecker delta
e	electron charge
ϵ	electromotive force
ϵ_{ijk}	completely antisymmetric rank-three tensor
η, η_t	magnetic diffusivity, turbulent magnetic diffusivity
$\mathcal{H}$	helicity
K	kinetic helicity
l_v	lengthscale for the velocity field
l_B	lengthscale for the magnetic field
l	lengthscale for the fluctuating part in mean-field theory
L	lengthscale for the average part in mean-field theory
λ_D	Debye length
ζ	lengthscale at which advection balances diffusion
Λ	plasma parameter
$\mathcal{L}$	characteristic lengthscale for the variation of the magnetic field
m_e	electron mass
m_p	proton mass
n_e	number density of electrons
n_i	number density of particles of species i
ν	viscosity

Ω	gyro frequency of electrons in a magnetic field
P	gas pressure
P_m	magnetic Prandtl number
r_L	gyro radius of electrons in a magnetic field
R_m	magnetic Reynolds number
R_e	Reynolds number
T_e	electron temperature
T_i	temperature of species i
τ	Ohmic diffusion timescale
τ_ζ	τ when B has structure of lengthscale ζ
$\vec{v}$	bulk velocity of the plasma
$v_\perp$	velocity perpendicular to $\vec{B}$
$\wedge$	vector product
V	volume
w_e	electron plasma frequency

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