# 1 Kinematics and Governing Equations

In this chapter, we quickly recapitulate the kinematics and governing equations for a fluid, which have been discussed in detail in Vol. I. We first state without proof some of the tensor identities discussed in Vol. I, which are of use in the subsequent discussion.

## 1.1 Tensor Identities

If $\boldsymbol{Q}$ is a proper orthogonal tensor, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are arbitrary vectors, then

$$\boldsymbol{Q}(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{Q}\boldsymbol{u})\times(\boldsymbol{Q}\boldsymbol{v}) \quad \text{(Eqn. (1.94) in Vol. I).} \tag{1.1}$$

If $\boldsymbol{W}$ is a skew-symmetric tensor, and $\boldsymbol{w}$ is its axial vector, then

$$\begin{aligned} w_i &= -\frac{1}{2}\epsilon_{ijk}W_{jk}, \\ W_{ij} &= -\epsilon_{ijk}w_k, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} &\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ &\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \\ &\epsilon_{ijk} = 0 \quad \text{otherwise.} \end{aligned}$$

If $\boldsymbol{u}$, $\boldsymbol{v}$ are arbitrary vector fields, and $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$, then

$$(\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times\boldsymbol{\omega}, \tag{1.3}$$

$$\boldsymbol{\nabla}^2\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\times\boldsymbol{\omega}, \tag{1.4}$$

$$\boldsymbol{\nabla}\times[(\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{u}] = (\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{\omega} + (\boldsymbol{\nabla}\boldsymbol{\omega})\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{\omega}, \tag{1.5}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v} + (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{u} - (\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{v} - (\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{u}. \tag{1.6}$$

Let $S$ represent the surface of a volume $V$, $\boldsymbol{n}$ represent the unit normal to the surface, $\phi$ a scalar field, $\boldsymbol{u}$ a vector field, and $\boldsymbol{T}$ a second-order tensor field all of which are continuous and differentiable. Then, we have

Divergence theorem (also known as Gauss' theorem)

$$\int_V \boldsymbol{\nabla}\phi\, dV = \int_S \phi\boldsymbol{n}\, dS. \tag{1.7}$$

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{u}\, dV = \int_S \boldsymbol{u} \cdot \boldsymbol{n}\, dS. \tag{1.8}$$

Stokes' theorem

Let $C$ be a contour, and $S$ be the area of any arbitrary surface enclosed by the contour $C$. Then,

$$\oint_C \boldsymbol{u} \cdot \boldsymbol{dx} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n}\, dS, \tag{1.9}$$

$$\oint_C \boldsymbol{u} \times \boldsymbol{dx} = \int_S \left[ (\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{n} - (\boldsymbol{\nabla}\boldsymbol{u})^T \boldsymbol{n} \right] dS. \tag{1.10}$$

## 1.2 Kinematics

### 1.2.1 Lagrangian and Eulerian descriptions

Consider a blob of fluid occupying a region in space $V_0$, and undergoing motion due to the action of some forces so as to occupy a different region of space $V(t)$, as shown in Fig. 1.1. The initial configuration $V_0$ can be taken to be reference configuration with respect to which the deformations in the current configuration $V(t)$ are measured. The choice of the reference configuration is arbitrary, and as a matter of convenience, the time is set at $t = 0$ at this configuration. As shown, a particle with a position vector $\boldsymbol{X}$ in the reference configuration $V_0$ occupies the position $\boldsymbol{x}$ in the current configuration $V(t)$. We shall

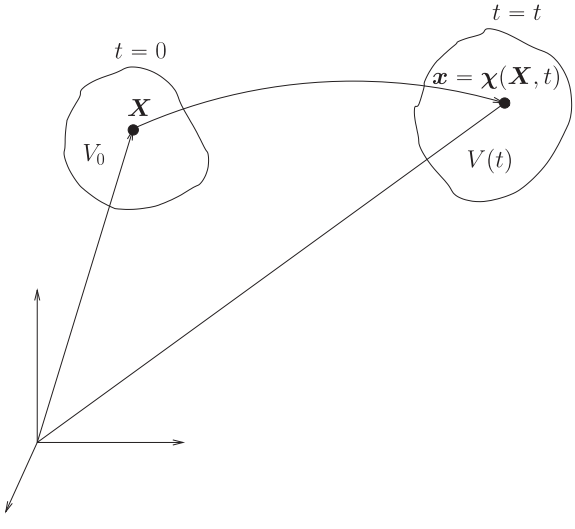


**Fig. 1.1** *Reference and current configurations.*

find it convenient to identify points of the body with the positions that they occupy in the reference configuration. Thus, points $\boldsymbol{X} \in V_0$ will often be called as *material points*.

The relation between $\boldsymbol{X}$ and $\boldsymbol{x}$ is expressed as

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t). \tag{1.11}$$

The coordinates $\boldsymbol{X}$ and $\boldsymbol{x}$ are known as the *material* and *spatial* coordinates, respectively. The mapping $\boldsymbol{\chi}$ taking $\boldsymbol{X}$ to $\boldsymbol{x}$ is assumed to be one-to-one and orientation-preserving. Physically, the restriction that the mapping characterizing the deformation be one-to-one corresponds to the requirement that two fluid particles in the reference configuration should not occupy the same position in the current configuration, and conversely, a fluid particle in the current configuration cannot occupy two or more positions in the current configuration. Due to the one-to-one nature of the mapping $\boldsymbol{\chi}$, we can invert it to obtain $\boldsymbol{X}$ as a function of $\boldsymbol{x}$ and $t$, i.e.,

$$\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t).$$

The formulation in terms of the material coordinates is called as the *Lagrangian* formulation, while the formulation in terms of the spatial coordinates is called the *Eulerian* formulation. The Lagrangian formulation is the one generally preferred in elasticity since the equations of equilibrium in the deformed configuration are expressed in terms of the spatial coordinates $\boldsymbol{x}$, which are themselves unknown. Hence, all the equations are formulated on the reference domain which is usually taken to be the initial unstressed state. On the other hand, the Eulerian formulation is preferred for problems in fluid mechanics, where we observe the flow in a fixed region of space such as a channel or wind-tunnel.

At each point in the domain, we define the deformation gradient $\boldsymbol{F}$ by

$$F_{ij} := \frac{\partial \chi_i}{\partial X_j} = \begin{bmatrix} \frac{\partial \chi_1}{\partial X_1} & \frac{\partial \chi_1}{\partial X_2} & \frac{\partial \chi_1}{\partial X_3} \\ \frac{\partial \chi_2}{\partial X_1} & \frac{\partial \chi_2}{\partial X_2} & \frac{\partial \chi_2}{\partial X_3} \\ \frac{\partial \chi_3}{\partial X_1} & \frac{\partial \chi_3}{\partial X_2} & \frac{\partial \chi_3}{\partial X_3} \end{bmatrix}.$$

At a given instant of time, a length element in the deformed configuration $\boldsymbol{dx}$ is related to a length element in the reference configuration $\boldsymbol{dX}$ by the following relationship:

$$\boldsymbol{dx} = \boldsymbol{F}\boldsymbol{dX}; \qquad dx_i = F_{ij} dX_j = \frac{\partial x_i}{\partial X_j} dX_j. \tag{1.12}$$

As discussed in Vol. I, we require the deformation to be orientation-preserving, i.e., (denoting $\det \boldsymbol{F}$ by $J$)

$$J > 0 \quad \forall \boldsymbol{X} \in V_0. \tag{1.13}$$

Given a material field $\phi(\boldsymbol{X}, t)$, the particle or material derivative of that field, denoted by $D\phi/Dt$, is defined as the partial derivative of $\phi$ with respect to time, i.e.,

$$\frac{D\phi}{Dt} := \left( \frac{\partial \phi}{\partial t} \right)_{\boldsymbol{X}}.$$

In the Lagrangian approach, the velocity and acceleration are defined as the first and second material derivatives, respectively, of the mapping $\chi$, i.e.,

$$\tilde{\boldsymbol{u}}(\boldsymbol{X},t) = \frac{D\boldsymbol{\chi}}{Dt} = \left(\frac{\partial \boldsymbol{\chi}}{\partial t}\right)_{\boldsymbol{X}},$$

$$\tilde{\boldsymbol{a}}(\boldsymbol{X},t) = \frac{D\tilde{\boldsymbol{u}}}{Dt} = \left(\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}\right)_{\boldsymbol{X}} = \left(\frac{\partial^2 \boldsymbol{\chi}}{\partial t^2}\right)_{\boldsymbol{X}}.$$

However, in fluid mechanics the problem with finding the velocities and accelerations from the above formulae is that, in general, at any given time $t$, we do not know the reference position $\boldsymbol{X}$ occupied by a particle now at $\boldsymbol{x}$. In such a case, the computation can be carried out using the Eulerian approach, in which the field quantities are now expressed as functions of the spatial position $\boldsymbol{x}$ and time. Since $\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x},t)$, the Eulerian description of the velocity is given by

$$\boldsymbol{u}(\boldsymbol{x},t) = \tilde{\boldsymbol{u}}(\boldsymbol{X},t) = \tilde{\boldsymbol{u}}(\boldsymbol{\chi}^{-1}(\boldsymbol{x},t),t).$$

In general, the functions $\tilde{\boldsymbol{u}}$ and $\boldsymbol{u}$ are different, though of course, their values at some material point $\boldsymbol{X}$, and the corresponding spatial point $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X},t)$, are the same. For an example, see Vol. I.

The expression for the acceleration using the spatial description of the velocity $\boldsymbol{u}(\boldsymbol{x},t)$, is given by

$$\boldsymbol{a}(\boldsymbol{x},t) = \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_{\boldsymbol{x}} + (\boldsymbol{\nabla}_x \boldsymbol{u})\boldsymbol{u}. \tag{1.14}$$

The above expression for the acceleration shows that one has to distinguish between the acceleration of a particle, and the local rate of change of velocity $(\partial \boldsymbol{u}/\partial t)_{\boldsymbol{x}}$ at a point. For example, in a steady flow, the local rate of change of velocity at any point is zero. However, this does not mean that the acceleration of a fluid particle passing through that point is zero, since its velocity can change as it moves to a different position. We shall henceforth write $(\partial/\partial t)_{\boldsymbol{x}}$ and $\boldsymbol{\nabla}_x$ simply as $\partial/\partial t$ and $\boldsymbol{\nabla}$, respectively.

One can compute the material derivative of any spatial field (also referred to as total, Eulerian or convective derivative), be it scalar, vector or tensor-valued, in a manner analogous to that above. For example, if $\phi(\boldsymbol{x},t)$ is a scalar-valued field, then its material derivative is given by

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_i}\frac{\partial \chi_i}{\partial t} = \frac{\partial \phi}{\partial t} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\phi). \tag{1.15}$$

### 1.2.2 Flow lines

The three kinds of flow trajectories commonly used to describe a flow are

1. *Pathlines:* A pathline gives the trajectory of a fluid particle as it flows. The initial position of the particle $\boldsymbol{X}$ is assumed to be fixed, and the trajectory is obtained as a function of time. If $\boldsymbol{x}(t) = \boldsymbol{\chi}(\boldsymbol{X}, t)$ denotes the trajectory of the particle then the governing differential equation for a pathline is given by
$$\frac{d\boldsymbol{x}(t)}{dt} = \left(\frac{\partial \boldsymbol{\chi}}{\partial t}\right)_{\boldsymbol{X}} = \boldsymbol{u}(\boldsymbol{x}, t), \tag{1.16}$$
which is solved subject to the initial condition $\boldsymbol{x}(0) = \boldsymbol{X}$.

2. *Streamlines:* Streamlines are lines whose tangent is everywhere parallel to the velocity vector at any given instant $t$. Since the velocity is a function of position and time, the streamlines can change from instant to instant. If $\boldsymbol{x}(s)$ represents the equation of a streamline as a function of the arc length $s$, the tangent to a streamline is given by $\boldsymbol{\lambda} = d\boldsymbol{x}/ds$. Since the velocity vector is parallel to the tangent vector at each point, the parametric equation of a streamline at an instant of time $t_0$ is given by
$$\frac{d\boldsymbol{x}(s)}{ds} = \boldsymbol{u}(\boldsymbol{x}(s), t_0). \tag{1.17}$$
In component form, the equation of a streamline is given by
$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3}. \tag{1.18}$$

3. *Streak-lines:* A streak-line is a line on which lie those fluid particles that all passed an injection point $\boldsymbol{x} = \boldsymbol{X}$ (like fluid through the tip of a syringe or smoke from a candle). The equation of a streakline is found by integrating equation (1.16) subject to the initial condition $\boldsymbol{x}|_{t=\tau} = \boldsymbol{X}$.

All the above flow lines are identical for steady (time-independent) flow since in that case we can write $\boldsymbol{u}(\boldsymbol{x}, t)$ simply as $\boldsymbol{u}(\boldsymbol{x})$, so that the parametric equations given by Eqns. (1.16) and (1.17) are the same, namely $d\boldsymbol{x}/dt = \boldsymbol{u}(\boldsymbol{x})$.

### 1.2.3 Analysis of deformation

In this section, we define the rate-of-deformation tensor (also known as rate-of-strain tensor), and vorticity tensors as a function of the velocity gradients.

#### Rate of deformation

The velocity gradient $\boldsymbol{L}$ is defined as the spatial gradient of the velocity. Thus,

$$\boldsymbol{L} := \boldsymbol{\nabla}_x \boldsymbol{u}.$$

Equation (1.14) can now be written as

$$\boldsymbol{a} = \frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{L}\boldsymbol{u}. \tag{1.19}$$

$\boldsymbol{L}$ can be split uniquely into a symmetric and antisymmetric part as

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \tag{1.20}$$

where $\boldsymbol{D}$ and $\boldsymbol{W}$ are referred to as the rate-of-deformation and vorticity tensors, respectively, and are given by

$$\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T); \qquad D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.21}$$

$$\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T); \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{1.22}$$

We have the following relations:

$$\frac{D\boldsymbol{F}}{Dt} = \left(\frac{\partial \boldsymbol{F}}{\partial t}\right)_{\boldsymbol{X}} = \boldsymbol{L}\boldsymbol{F}, \tag{1.23a}$$

$$\frac{D\boldsymbol{F}^{-1}}{Dt} = \left(\frac{\partial \boldsymbol{F}^{-1}}{\partial t}\right)_{\boldsymbol{X}} = -\boldsymbol{F}^{-1}\boldsymbol{L}, \tag{1.23b}$$

$$\frac{D}{Dt}\left(\frac{D\boldsymbol{F}}{Dt}\right) = (\boldsymbol{\nabla}\boldsymbol{a})\boldsymbol{F}, \tag{1.23c}$$

$$\frac{D}{Dt}\left(\boldsymbol{F}^T\boldsymbol{W}\boldsymbol{F}\right) = \boldsymbol{F}^T\left[\frac{D\boldsymbol{W}}{Dt} + \boldsymbol{W}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{W}\right]\boldsymbol{F}, \tag{1.23d}$$

$$\frac{1}{J}\frac{DJ}{Dt} = \frac{1}{J}\left(\frac{\partial J}{\partial t}\right)_{\boldsymbol{X}} = \operatorname{tr}\boldsymbol{L} = \boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{1.23e}$$

From Eqns. (1.23a) and (1.23c), it follows that

$$\boldsymbol{\nabla}\boldsymbol{a} = \frac{D\boldsymbol{L}}{Dt} + \boldsymbol{L}^2. \tag{1.24}$$

Further, by taking the trace of the above relation, we get

$$\frac{D(\boldsymbol{\nabla}\cdot\boldsymbol{u})}{Dt} = \boldsymbol{\nabla}\cdot\boldsymbol{a} - \boldsymbol{L} : \boldsymbol{L}^T. \tag{1.25}$$

The above relation can also be derived by taking the divergence of Eqn. (1.19) (see Problem 9, Chapter 2, Vol. I)

In the fluid-mechanics literature, the rate-of-deformation tensor is often referred to as the rate-of-strain tensor. The terminology 'rate-of-strain' seems to imply that $\boldsymbol{D}$ is actually the rate of some strain quantity. In reality, it is neither the rate of the small-strain tensor, nor the rate of the Lagrangian or the Eulerian strain tensors, even in an approximate sense [62].

Hence, we shall desist from using the terminology 'rate-of-strain' for $\boldsymbol{D}$. It is important to note that the expression for $\boldsymbol{D}$ given by Eqn. (1.21) is valid for *arbitrary* deformations, since no assumption has been made about their magnitude. We now justify the terminology 'rate-of-deformation' used for $\boldsymbol{D}$.

The square of a length of fluid element in the current configuration is given by $(ds)^2 = \boldsymbol{dx} \cdot \boldsymbol{dx}$. Hence, the material rate of change of $(ds)^2$ is given by

$$\begin{aligned}
\frac{D}{Dt}(ds)^2 &= 2\boldsymbol{dx} \cdot \frac{D}{Dt}(\boldsymbol{dx}) \\
&= 2\boldsymbol{dx} \cdot \frac{D}{Dt}(\boldsymbol{F}\boldsymbol{dX}) && \text{(by Eqn. (1.12))} \\
&= 2\boldsymbol{dx} \cdot \frac{D\boldsymbol{F}}{Dt}\boldsymbol{dX} && \text{(since } \boldsymbol{dX} \text{ is a constant)} \\
&= 2\boldsymbol{dx} \cdot \boldsymbol{L}\boldsymbol{F}\boldsymbol{dX} && \text{(by Eqn. (1.23a))} \\
&= 2\boldsymbol{dx} \cdot \boldsymbol{L}\boldsymbol{dx} && \text{(by Eqn. (1.12))} \\
&= 2\boldsymbol{dx} \cdot (\boldsymbol{D} + \boldsymbol{W})\boldsymbol{dx} && \text{(by Eqn. (1.20))} \\
&= 2\boldsymbol{dx} \cdot \boldsymbol{D}\boldsymbol{dx}, && (1.26)
\end{aligned}$$

where in the last step we have used the antisymmetry of $\boldsymbol{W}$ as follows:

$$\boldsymbol{dx} \cdot \boldsymbol{W}\boldsymbol{dx} = \boldsymbol{W}^T\boldsymbol{dx} \cdot \boldsymbol{dx} = -\boldsymbol{dx} \cdot \boldsymbol{W}\boldsymbol{dx},$$

which implies that $\boldsymbol{dx} \cdot \boldsymbol{W}\boldsymbol{dx} = 0$.

The vorticity vector is defined by

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}. \tag{1.27}$$

We now show that $\boldsymbol{\omega}/2$ is the axial vector of $\boldsymbol{W}$. It then follows from Eqns. (1.2) that

$$\begin{aligned}
\omega_i &= -\epsilon_{ijk} W_{jk}, \\
W_{ij} &= -\frac{1}{2}\epsilon_{ijk}\omega_k.
\end{aligned} \tag{1.28}$$

Using $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$, we have

$$\begin{aligned}
(\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{v} &= \epsilon_{ijk}\epsilon_{jmn}\frac{\partial u_n}{\partial x_m} v_k \boldsymbol{e}_i \\
&= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})\frac{\partial u_n}{\partial x_m} v_k \boldsymbol{e}_i \\
&= \left[\frac{\partial u_i}{\partial x_k} v_k - \frac{\partial u_k}{\partial x_i} v_k\right] \boldsymbol{e}_i \\
&= [\boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T]\boldsymbol{v} \\
&= 2\boldsymbol{W}\boldsymbol{v} \quad \forall \boldsymbol{v} \in V,
\end{aligned}$$

which is the desired result. Physically speaking, the vorticity vector represents the local rotation of an element. The volume elements in the deformed and undeformed configurations are related as

$$dV = JdV_0. \tag{1.29}$$

### 1.2.4 Vortex lines and circulation

A *vortex line* is a line to which the vorticity vector is tangent everywhere. By following a procedure similar to the derivation of the equation for a streamline, we find the equation for a vortex line to be

$$\frac{dx_1}{\omega_1} = \frac{dx_2}{\omega_2} = \frac{dx_3}{\omega_3}.$$

A *vortex tube* is a tube with a wall composed of vortex lines. The *circulation* contained within a closed contour $C$ is defined by

$$\Gamma := \oint_C \boldsymbol{u} \cdot \boldsymbol{dx} = \oint_C \boldsymbol{u} \cdot \boldsymbol{\lambda}\, ds, \tag{1.30}$$

where $\boldsymbol{dx}$ represents an element of the contour, and $\boldsymbol{\lambda} = d\boldsymbol{x}/ds$ represents the unit tangent to the contour. Applying Stokes' theorem (Eqn. (1.9)) to Eqn. (1.30) we get

$$\Gamma = \oint_C \boldsymbol{u} \cdot \boldsymbol{dx} = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS. \tag{1.31}$$

Thus, the circulation represents the flux of vorticity through $S$. Note that if $\boldsymbol{\omega} = \boldsymbol{0}$, then $\Gamma = 0$. Flows for which $\boldsymbol{\omega} = \boldsymbol{0}$ are called *irrotational*. One can prove that $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$ if and only if there exists a scalar potential $\phi$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. Since $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0$, for a closed surface $S_c$ we have

$$\int_{S_c} \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\omega}\, dV = 0. \tag{1.32}$$

Consequently, we conclude that for a closed surface, vortex lines cannot terminate in the interior of the surface, i.e., the net flux of the vorticity through $S_c$ is zero. This fact is used to prove that the circulations along two closed contours which enclose different cross sections on a vortex tube are identical, as follows. Consider the section of a vortex tube shown in Fig. 1.2. The surfaces $S$, $A_1$ and $A_2$ constitute the total surface $S_c$ of the section. $C_1$ and $C_2$ represent the contours of $A_1$ and $A_2$, respectively. The circulation associated with $C_1$ is given by $\Gamma_1 = \oint_{C_1} \boldsymbol{u} \cdot \boldsymbol{dx}$, and that associated with $C_2$ is $\Gamma_2 = \oint_{C_2} \boldsymbol{u} \cdot \boldsymbol{dx}$. Since $\boldsymbol{\omega}$ is perpendicular to the normal $\boldsymbol{n}$ on the surface of a vortex tube, $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ on $S$. Hence, from Eqn. (1.32), we have

$$\int_{A_1} \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS + \int_{A_2} \boldsymbol{\omega} \cdot \boldsymbol{n}\, dS = 0.$$

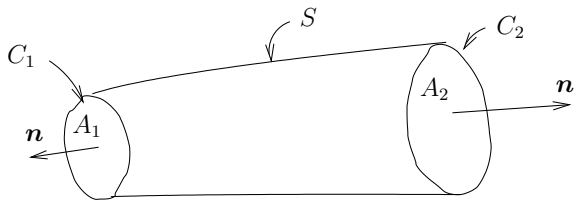


Fig. 1.2 *Section of a vortex tube.*

Using Eqn. (1.9), and assuming a counter-clockwise circulation as positive, this can be written as

$$\oint_{C_1} \boldsymbol{u} \cdot \boldsymbol{dx} - \oint_{C_2} \boldsymbol{u} \cdot \boldsymbol{dx} = 0,$$

which proves that $\Gamma_1 = \Gamma_2$.

We now present a summary of the transport theorems and the governing equations based on the principles of conservation of mass, balance of linear and angular momentum, and the laws of thermodynamics; see Vol. I for the derivations.

## 1.3 Governing Equations

The following transport theorems are used in deriving the governing equations:

**Theorem 1.3.1** (Transport theorem-I). *Let $f(\boldsymbol{x}, t)$ be a continuous and differentiable scalar valued function, and let $V(t)$ be the material volume (i.e., a volume comprising of a fixed set of particles, and moving with the medium). Then, due to integration over the spatial coordinates, $\int_{V(t)} f(\boldsymbol{x}, t)\, dV$ is a function of time alone; its time derivative is given by*

$$\frac{d}{dt} \int_{V(t)} f(\boldsymbol{x}, t)\, dV = \int_{V(t)} \frac{\partial f}{\partial t}\, dV + \int_{S(t)} f(\boldsymbol{u} \cdot \boldsymbol{n})\, dS. \tag{1.33}$$

If $\boldsymbol{w}(\boldsymbol{x}, t)$ is a vector-valued function, then we apply Eqn. (1.33) to each component of $\boldsymbol{w}$ to get

$$\frac{d}{dt} \int_{V(t)} \boldsymbol{w}(\boldsymbol{x}, t)\, dV = \int_{V(t)} \frac{\partial \boldsymbol{w}}{\partial t}\, dV + \int_{S(t)} \boldsymbol{w}(\boldsymbol{u} \cdot \boldsymbol{n})\, dS. \tag{1.34}$$

A control volume $V$ is usually (but not always) a fixed region of space where we observe various flow quantities. Since, in many instances, it is possible to compute forces or the power generated simply by having the appropriate information at the control surface, the

control volume form of the governing equation is very useful. The weakness of this approach, however, is that we do not obtain the details of the various fields within the control volume.

Unlike the material volume $V(t)$ which comprises the same set of particles as it evolves, the control volume $V$ will, in general, have different sets of particles at different times, and even the mass of material enclosed by the control volume may change as, say, in the case of a control volume enclosing a rocket. If $S$ denotes the control surface, and $\boldsymbol{u}_{\text{rel}}$ denotes the relative velocity of the fluid with respect to $S$, then we have the important relation

$$\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x},t)\,dV = \frac{d}{dt}\int_{V} f(\boldsymbol{x},t)\,dV + \int_{S} f(\boldsymbol{u}_{\text{rel}}\cdot\boldsymbol{n})\,dS. \tag{1.35}$$

For a vector-valued function $\boldsymbol{w}(\boldsymbol{x},t)$, the corresponding relation is

$$\frac{d}{dt}\int_{V(t)} \boldsymbol{w}(\boldsymbol{x},t)\,dV = \frac{d}{dt}\int_{V} \boldsymbol{w}(\boldsymbol{x},t)\,dV + \int_{S} \boldsymbol{w}(\boldsymbol{u}_{\text{rel}}\cdot\boldsymbol{n})\,dS. \tag{1.36}$$

Let $\boldsymbol{u}_{\text{cs}}$ denote the velocity of the control surface. If

$$\int_{S_t} f(\boldsymbol{u}_{\text{cs}}\cdot\boldsymbol{n})\,dS = 0; \quad \int_{S_t} \boldsymbol{w}(\boldsymbol{u}_{\text{cs}}\cdot\boldsymbol{n})\,dS = \boldsymbol{0}, \tag{1.37}$$

then the time derivative in the first terms on the right-hand side of Eqns. (1.35) and (1.36) can be taken inside the integral sign, i.e.,

$$\frac{d}{dt}\int_{V} f(\boldsymbol{x},t)\,dV = \int_{V}\left(\frac{\partial f}{\partial t}\right)_{\boldsymbol{x}} dV; \quad \frac{d}{dt}\int_{V} \boldsymbol{w}(\boldsymbol{x},t)\,dV = \int_{V}\left(\frac{\partial \boldsymbol{w}}{\partial t}\right)_{\boldsymbol{x}} dV. \tag{1.38}$$

Sufficient (but not necessary) conditions under which Eqn. (1.37) (and hence, Eqn. (1.38)) holds are

1. the control volume is stationary, i.e., $\boldsymbol{u}_{\text{cs}} = \boldsymbol{0}$, or
2. the control volume is undergoing rigid motion, *and* $f$ (or $\boldsymbol{w}$) is a function of time alone,

For the subsequent development, we define the density field as

$$\rho(\boldsymbol{x},t) = \lim_{\Delta V\to 0}\frac{\Delta m}{\Delta V},$$

where $\Delta V$ is a small volume with fluid of mass $\Delta m$, surrounding the point with position vector $\boldsymbol{x}$. We now state the second transport theorem.

**Theorem 1.3.2** (Transport theorem-II). *Let $f(\boldsymbol{x},t)$ be as in transport theorem I. Then, $\int_{V(t)}\rho f(\boldsymbol{x},t)\,dV$ due to integration over the spatial coordinates, is a function of time alone, and its time derivative is given by*

$$\frac{d}{dt}\int_{V(t)}\rho f(\boldsymbol{x},t)\,dV = \int_{V(t)}\rho\frac{Df}{Dt}\,dV = \int_{V(t)}\rho\left\{\frac{\partial f}{\partial t} + \boldsymbol{u}\cdot(\boldsymbol{\nabla} f)\right\}dV. \tag{1.39}$$