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Introduction

Classical continuum physics deals with media without a visible microstructure. That is, the scale of observation is large compared to the molecular scale, but small relative to other heterogeneities within the system. More modern continuum theories consider more directly the influence of microstructures; among them are micromorphic, mixture, and nonlocal theories. On the smallest scale, individual molecules are observed. Statistical mechanical theories and some micromorphic field theories may be applicable on this scale. On a slightly larger scale, the material body appears locally uniform with no distinct microstructure. This is the scale of observation on which classical continuum theories apply. On yet a larger scale of observation, large heterogeneities in space and/or time are evident. Such heterogeneities are well characterized by the solution of continuum mechanics problems at these scales.

From the atomic point of view, a macroscopic sample of matter is an agglomerate of an enormous number of nuclei and electrons. A complete mathematical description of a sample consists of the specification of suitable coordinates for each nucleus and electron; the number of such coordinates is enormous considering the magnitude of Avogadro's number of 6.0221×10^{23} mol^{-1} which gives us the number of molecules in one mole of a substance.

In contrast to the atomistic description, only a few parameters are required to describe the system macroscopically. The key to this reduction is the slowness and large scale of macroscopic measurements in comparison to the speed of atomic motions (typically of the order of 10^{-15} s) and atomic distance scales (typically of the order of 10^{-10} m). For example, some of our fastest macroscopic measurements are of the order of 10^{-6} s. Consequently, macroscopic measurements sense only averages of the atomic coordinates. The mathematical process of averaging eliminates coordinates and thus reduces the level of description in going from the atomic to the macroscopic level.

Of the enormous number of atomic coordinates, a very few, with unique symmetry properties, survive the statistical averaging. Certain of these are mechanical in nature (e.g., volume, shape, and components of elastic strain), others are thermal in nature (e.g., temperature and internal energy), or electrical/magnetical in nature (e.g., electric and magnetic dipole moments). The subject of mechanics (e.g., elasticity and fluid mechanics) is the study of one set of surviving coordinates, the subject of thermal sciences (e.g., thermodynamics and heat transfer) is the

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study of another set of surviving coordinates, and the subject of electricity and magnetism is the study of still another set of coordinates. In general, all these sets of coordinates are coupled, and the study of continuum mechanics provides the framework to study the coupling between these coordinates.

For many materials the behavior of large samples can be studied without recourse to the details of the atomic level structure. We can describe fluids, solids, glasses, bio-materials, mixtures, etc., by making use of the framework provided by continuum mechanics.

1.1 Continuum mechanics

Continuum mechanics is the study of the macroscopic consequences of the large number of atomic coordinates, which, by virtue of statistical averaging, do not appear explicitly in the macroscopic description of a system. It is a branch of physics that deals with materials. The fact that matter is made of atoms and that it commonly has some sort of heterogeneous microstructure is mostly ignored in the simplifying approximation that physical quantities, such as mass, momentum, and energy, can be handled in the infinitesimal limit. For most materials, this is possible as long as the characteristic length scale is far larger than 10^{-9} m and the characteristic speed is much less than the speed of light (3×10^8 m/s). If the length scale is of the order of 10^{-9} m or less, then quantum mechanics applies. If the speed is near the speed of light, then relativistic mechanics applies. If the length scale is of the order of 10^{-9} or less and the speed is near that of light, then quantum field theory applies.

What are the consequences of the existence of the "hidden" atomic motion? Recall that in mechanics, thermal sciences, and electricity and magnetism we are much concerned with the concept of energy. Energy transferred to a mechanical mode of a system is called *mechanical work* $\delta\mathcal{W}$. Similarly, energy can be transferred to an electrical mode of the system. Mechanical work is typified by the term $-p\,dV$ (p is pressure and V is volume), and *electrical work* is typified by the term $-\mathcal{E}\,d\mathcal{P}$ ($\mathcal{E}$ is the electric field and $\mathcal{P}$ is the electric dipole moment). It is equally possible to transfer energy to the hidden atomic modes of motion as well as to those which happen to be macroscopically observable. Energy transfer to the hidden atomic modes is called *heat*. The energy residing in the hidden atomic motions we call *internal energy*. Heat transfer and internal energy are typified by terms such as $\delta\mathcal{Q}$ and dU.

Continuum mechanics is very general; it applies to complicated systems with mechanical, thermal, and electrical/magnetical properties. In this book, we will focus on mechanical and thermal properties of materials, keeping in mind that this is not a limitation of continuum mechanics theory. Differential equations are employed in solving problems in continuum mechanics. Some of these differential equations are specific to the materials being investigated, while others capture fundamental physical laws, such as conservation of mass or conservation of momentum.

The physical laws of a material's response to forces do not depend on the coordinate system in which they are observed. Continuum mechanics is thus described by tensors, which are mathematical objects that are independent of a coordinate system. Such tensors can be expressed in coordinate systems for computational convenience.

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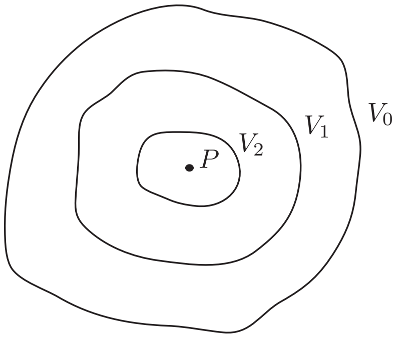

Figure 1.1: Limit at a point P.

1.2 Continuum

A *continuum* is a classical concept derived from mathematics:

a) the real number system is a continuum;

b) *time* can be represented by a real number system;

c) *three-dimensional space* can be represented by three real number systems;

d) time-space together is identified as a four-dimensional continuum.

A material continuum is characterized by quantities such as mass, momentum, energy, and state variables.

Matter, as measured by its mass m, is assumed to have a continuous distribution in space. A certain amount of mass occupies a definite volume V. As illustrated in Fig. 1.1, we define the mass density at an arbitrary point P by

$$\rho(P) = \lim_{\substack{n\to\infty \\ V_n\to 0}} \frac{m_n}{V_n}, \tag{1.1}$$

where m_n is the mass contained in the averaging volume V_n.

Since the averaging volume must be sufficiently larger than molecular scales, to conform to the real world, we take the definition of the density of the material at P with an acceptable variability $\epsilon > 0$ in a defining limit volume $\delta > 0$:

$$\lim_{\substack{n\to\infty \\ V_n\to\delta\ll 1}} \left| \frac{\rho(P)}{m_n/V_n} - 1 \right| < \epsilon \ll 1. \tag{1.2}$$

It is our responsibility to make sure that δ is sufficiently large and ϵ sufficiently small for the concepts of a continuum to make sense. For example, δ should be large enough in the four-dimensional time-space continuum to include a sufficiently large number of molecules so that the number of molecules entering or leaving δ is such as to lead to ϵ sufficiently small. Similarly, we define densities of momentum and energy. For vector quantities, the definition applies to each component individually. Note that in general the size of the limit volume δ for a fixed acceptable variability ϵ is different for different physical quantities. Thus, again, it is

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our responsibility to understand that the continuum description only makes sense when describing average properties at scales larger than the largest δ among all quantities that we are interested in describing within the acceptable variability ϵ.

Continuum mechanics ignores all the fine detail of atomic and molecular (or particle) level structure and assumes that

- the highly discontinuous structure of real materials can be replaced by a smoothed hypothetical continuum;

- every portion of the continuum, however small, exhibits the macroscopic physical properties of the bulk material.

In any branch of continuum mechanics, the field variables (i.e., density, displacement, and velocity) are conceptual constructs. They are taken to be defined at all points of the imagined continuum and their values are calculated via axiomatic rules of procedure.

The continuum model breaks down over distances comparable to interatomic spacing (in solids about 10^{-10} m). Nonetheless, the *average* of a field variable over a small but *finite* region is meaningful. Such an average can, in principle, be compared directly to its nominal counterpart found by experiment, which will itself represent an average of a kind taken over a region containing many atoms, because of the finite physical size of any measuring probe. For solids, the continuum model is valid in this sense down to a scale of order 10^{-8} m which is the side of a cube containing a million or so atoms. Further, when field variables change slowly with position at a microscopic level $\sim 10^{-6}$ m, their averages over such volumes (10^{-20} m^3 say) differ insignificantly from their centroidal values. In this case, pointwise values can be compared directly to observations. Such behaviors are illustrated in Fig. 1.2 for the mass density at point P as a function of the size of the averaging volume.

Within the continuum we take the behavior to be determined by balance laws for mass, linear momentum, angular momentum, energy, and the second law of thermodynamics. The continuum hypothesis enables us to apply these laws on a local as well as a global scale.

1.3 Mechanics

Classical mechanics is the study of the motion and deformation changes in a body composed of matter due to the action of forces. It is often referred to as *Newtonian mechanics* after Newton and his laws of motion. Classical mechanics is subdivided into statics (which models objects at rest), kinematics (which models objects in motion), and dynamics (which models objects subjected to forces). In continuum mechanics, we deal with all three aspects that are based on the concepts of time, space, and forces. To understand the concept of forces, knowledge is needed from all branches of engineering, physics, chemistry, and biology.

Classical mechanics produces very accurate results within the domain of everyday experience. It is superseded by relativistic mechanics for systems moving at large velocities (near the speed of light), quantum mechanics for systems at small spatial scales (atomic or subatomic scales), and relativistic quantum field theory for systems with both properties. Nevertheless, classical mechanics is still very

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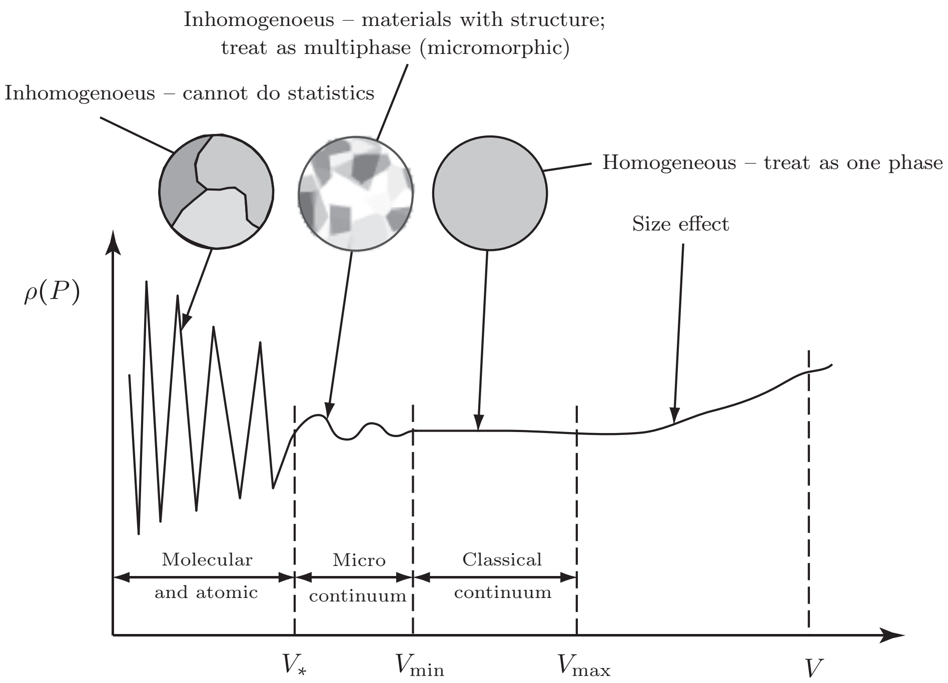

Figure 1.2: Density limit with acceptable variability at a point P as a function of averaging volume.

useful, because (i) it is much simpler and easier to apply than these other theories, and (ii) it has a very large range of approximate validity. Classical mechanics can be used to describe the motion of human-sized objects (i.e., tops and baseballs), many astronomical objects (i.e., planets and galaxies), and certain microscopic objects (i.e., sand grains and organic molecules.)

1.3.1 Deformation and strain

If we take a solid cube and subject it to some deformation, the most obvious change in external characteristics will be a modification of the shape.

The specification of the deformation is thus a geometrical problem and may be carried out from two different viewpoints: relate the deformation

1. with respect to the undeformed state (Lagrangian), or

2. with respect to the deformed state (Eulerian).

Locally, the mapping from the deformed to the undeformed state can be assumed to be linear and described by a differential relation, which is a combination of pure stretch (a rescaling of each coordinate) and a pure rotation.

The mechanical effects of the deformation are confined to the stretch and it is convenient to characterize this by a *strain* measure. For example, for a wire under

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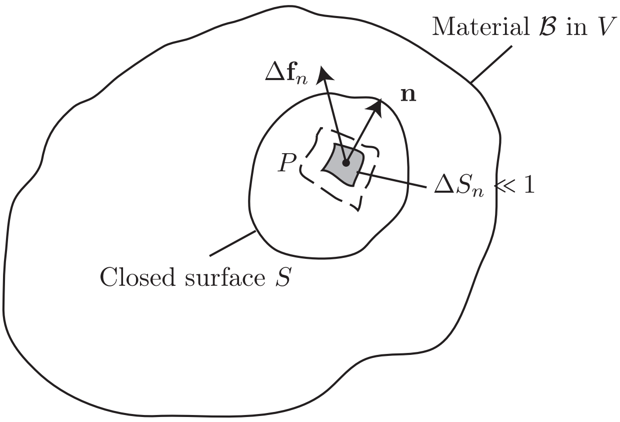

Figure 1.3: Traction with acceptable variability at a point P.

load the strain would be the relative extension, i.e.,

$$\text{linear strain} = \frac{\text{change in length}}{\text{initial length}}.$$

The generalization of this idea requires us to introduce a strain tensor at each point of the continuum.

1.3.2 Stress field

Stress is a measure of force intensity or density. As illustrated in Fig. 1.3, the traction or stress vector $\mathbf{t}$ at an arbitrary point P on a surface with normal vector $\mathbf{n}$ with an acceptable variability $\epsilon > 0$ is defined by

$$\lim_{\substack{n\to\infty \\ \Delta S_n \to \alpha \ll 1}} \left| \frac{\mathbf{t}(P,\mathbf{n})}{\Delta\mathbf{f}_n/\Delta S_n} - 1 \right| < \epsilon \ll 1, \tag{1.3}$$

where $\alpha > 0$ is sufficiently small, or more simply

$$\mathbf{t}(P,\mathbf{n}) = \frac{d\mathbf{f}}{dS}.$$

Within a deformed continuum there will be a force system acting. If we were able to cut the continuum in the neighborhood of a point P as illustrated in Fig. 1.4, we would find a force acting on the cut surface, which would depend on the inclination of the surface and is not necessarily perpendicular to the surface. This force system can be described by introducing a stress tensor $\boldsymbol{\sigma}$ at each point whose components describe the loading characteristics.

1.4 Thermodynamics

Thermodynamics is the physics of energy, heat, work, entropy, and the spontaneity of processes.

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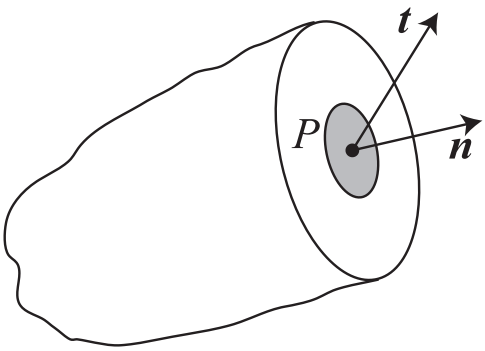

Figure 1.4: Stress vector at a point P.

While dealing with processes in which systems exchange matter or energy, classical thermodynamics is not concerned with the rate at which such processes take place, termed kinetics. For this reason, the use of the term *thermodynamics* usually refers to equilibrium thermodynamics. In this connection, a central concept in thermodynamics is that of quasistatic processes, which are idealized *infinitely slow* processes. Because thermodynamics is not concerned with the concept of time, it has been suggested that a better name for equilibrium thermodynamics would have been *thermostatics*. Time-dependent thermodynamic processes are studied by *nonequilibrium thermodynamics*. In continuum mechanics, we deal with both equilibrium and non-equilibrium thermodynamics.

Thermodynamic laws are of very general validity, and they do not depend on the details of the interactions or the systems being studied. This means they can be applied to systems about which one knows nothing other than the balance of energy and matter transfer between them and the environment.

The quantities that set thermostatics apart from classical particle mechanics are temperature and entropy. The significance of entropy can be illustrated as follows. Consider a flowing fluid. The fluid molecules possess kinetic energy which can be broken into two components, a part which is ordered and contributes to the bulk velocity, and another part which is random. The ordered energy is similar to the macroscopic kinetic energy of particle mechanics, and is mechanical in form. It is capable of being converted to work. Extraction of the ordered kinetic energy would leave only the random (thermal) energy in the fluid. The random component of the energy would contribute nothing to the work, as molecules would impact with such forces so as to cancel each other. Theoretically, one could extract all the ordered energy from the fluid leaving only the random energy. Now suppose that rather than extracting the organized energy, we somehow bring the convective fluid to a stop. The total energy of the fluid would remain unchanged, but there would no longer be any ordered component. All the kinetic energy of the molecules is now coming from random motions, and any attempt to convert this energy to work is fruitless. Entropy is a measure of the randomness, or of the energy's inability to do work – except through transfer of randomness from one body to another.

The random thermal energy will not freely convert back to mechanical form. That is, the likelihood that the molecules will realign to travel in some preferred direction is extremely small. Thus, since entropy is a measure of the randomness,

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it will not decrease without some external interactions. The only way one can decrease molecular randomness is to transfer some of this randomness to another body, and thereby increasing the randomness (entropy) of the other body. Thus, transfer of thermal energy from one body to another effectively transfers entropy. The transfer of thermal energy (randomness) as heat in this fashion is the only known way by which it is possible to reduce a body's entropy.

1.5 Constitutive theory

The specification of the stress and strain states of a body is insufficient to describe its full behavior; we need in addition to link these two fields.

This is achieved by introducing a *constitutive relation*, which prescribes the response of the continuum to arbitrary loading and thus defines the connection between the stress and strain tensors for the particular material.

At best a mathematical expression provides an approximation to the actual behavior of the material, but as we shall see we can simulate the behavior of a wide class of media.

In general, we think of materials as existing in either a solid or fluid state. The distinction between solid and fluid matter is relative; it depends on time scales over which the material deforms. In turn, we can view a solid as either hard or soft. Hard solids tend to respond elastically to an applied force, they have large acoustic speeds, their energy character is enthalpic, they tend to be anisotropic, they usually rupture upon yielding, and they retain perfect memory of only their initial state. On the other hand, soft solids tend to be dissipative, have a low acoustic speed, their energy behavior is entropic, they tend to be isotropic, they fail through plastic deformation (fluid like), and they behave as viscous on a short time scale and elastic on a long time scale. Fluids, in turn, can be classified as either isotropic or anisotropic. Isotropic fluids can exhibit time scale effects. In general, they respond elastically at short times and viscous at long times. Anisotropic fluids, such as liquid crystals, behave solid-like and exhibit elastic behavior in some directions.

1.5.1 Solids

To get an idea of the behavior of solids, we consider extension of a wire under loading. The tensile stress $\boldsymbol{\sigma}$ and tensile strain $\mathbf{e}$ are then typically related. A typical stress-strain curve is illustrated in Fig. 1.5.

a) Elasticity: If the wire returns to its original configuration when the load is removed, the behavior is said to be elastic.

   i) for linear elasticity $\boldsymbol{\sigma} = E\,\mathbf{e}$ – called Hooke's law and is usually valid for small strains (E is the elastic modulus);

   ii) for nonlinear elasticity $\boldsymbol{\sigma} = \mathbf{f}(\mathbf{e})$ – it is important for rubber-like materials.

b) Plasticity: Once the yield point is exceeded, permanent deformation occurs and there is no unique stress-strain curve, but a unique $d\boldsymbol{\sigma}$-$d\mathbf{e}$ relation. Due to microscopic processes, the yield stress rises with $\boldsymbol{\sigma}$ (work hardening).

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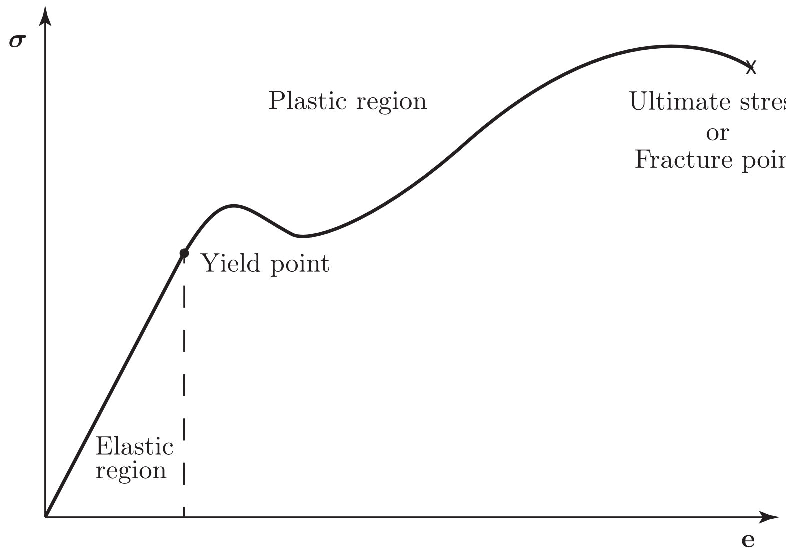

Figure 1.5: A typical stress-strain curve.

c) Viscoelasticity (rate-dependent behavior): Materials may creep and show slow long-term deformation, e.g., plastics and metals at elevated temperatures. Simple models of viscoelasticity are

   i) Maxwell model:

$$\dot{\boldsymbol{\sigma}} + \frac{E}{\mu}\boldsymbol{\sigma} = E\,\dot{\mathbf{e}}$$

   which allows for instantaneous elasticity and represents a crude description of a fluid (μ is the viscosity of the material).

   ii) Kelvin–Voigt model:

$$\boldsymbol{\sigma} = E\,\mathbf{e} + \mu\,\dot{\mathbf{e}}$$

   which displays long-term elasticity.

More complex models can be written down, but all have the same characteristic of depending on the time history of deformation.

1.5.2 Fluids

The simplest constitutive equation encountered in continuum mechanics is that of an ideal fluid:

$$\boldsymbol{\sigma} = -p(\rho, T)\,\mathbf{1},$$

where ρ is the density, T is the absolute temperature, the pressure field p is isotropic and depends on density and temperature, and $\mathbf{1}$ is the unit tensor. If the fluid is incompressible, ρ is a constant. The next level of complication is to allow the

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stress to depend on the flow of the fluid. The simplest such form, a Newtonian viscous fluid, includes a linear dependence on strain rate

$$\boldsymbol{\sigma} = -p(\rho, T)\ \mathbf{1} + \mu(\rho, T)\ \dot{\mathbf{e}}.$$

The quantity μ is called the *shear viscosity*.

1.6 Pioneers of continuum mechanics

The study of continuum mechanics originated from the works of James and John Bernoulli, Euler, and Cauchy. The field remained stagnant for a very long period of time after them. It was only after World War II that interest in the field was renewed. The modern field of continuum mechanics is the result of pioneering works from Truesdell, Noll, Toupin, Rivlin, Coleman, Ericksen, Müller, Eringen, Gurtin, and Liu, among others. Clifford Truesdell is considered the father of modern continuum mechanics.

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