PART I

BACKGROUND

Cambridge University Press
978-1-107-08585-5 - Introduction to the AdS/CFT Correspondence
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Excerpt
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1 Elements of quantum field theory and gauge theory

In this chapter, I review some useful issues about quantum field theory, assuming nevertheless that the reader has seen them before. It will also help to set up the notation and conventions.

1.1 Note on conventions

Throughout this book, I use theorist's conventions, with $\hbar = c = 1$. If we need to reintroduce them, we can use dimensional analysis. In these conventions, there is only one dimensionful unit, $mass = 1/length = energy = 1/time = \dots$ and when I speak of dimension of a quantity I refer to mass dimension, e.g. the mass dimension of d^4x, $[d^4x]$, is -4.

For the Minkowski metric $\eta^{\mu\nu}$ I use the signature convention mostly plus, thus for instance in 3+1 dimensions the signature will be $(-+++)$, giving $\eta^{\mu\nu} = diag(-1,+1,+1,+1)$. This convention is natural in order to make heavy use of the Euclidean formulation of quantum field theory and to relate to Minkowski space via Wick rotation.

Also, in this book we use *Einstein's summation convention*, i.e. indices that are repeated are summed over. Moreover, the indices summed over will be one up and one down, unless we are in Euclidean space, where up and down indices are the same.

1.2 The Feynman path integral and Feynman diagrams

To exemplify the basic concepts of quantum field theory, and the Feynman diagrammatic expansion, I use the simplest possible example, of a scalar field. A scalar field is a field that under a Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.1}$$

transforms as

$$\phi'(x'^\mu) = \phi(x^\mu). \tag{1.2}$$

We will deal with relativistic field theories, which are also *local*, which means that the action is an integral over functions defined at one point, of the type

$$S = \int L dt = \int d^4 x \mathcal{L}(\phi, \partial_\mu \phi). \tag{1.3}$$

Here L is the Lagrangean, whereas $\mathcal{L}(\phi(\vec{x}, t), \partial_\mu \phi(\vec{x}, t))$ is the *Lagrangean density*, that often times by an abuse of notation is also called Lagrangean.

Classically, one varies this action with respect to $\phi(x)$ to give the classical equations of motion for $\phi(x)$,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right]. \tag{1.4}$$

Quantum mechanically, the field $\phi(x)$ is not observable anymore, and instead one must use the vacuum expectation value (VEV) of the scalar field quantum operator instead, which is given as a "path integral" in Minkowski space,

$$\langle 0 | \hat{\phi}(x_1) | 0 \rangle = \int \mathcal{D}\phi e^{iS[\phi]} \phi(x_1). \tag{1.5}$$

Here the symbol $\int \mathcal{D}\phi$ represents a discretization of a spacetime path $\phi(x_1^\mu) \to \phi(x_2^\mu)$, followed by integration over the field value at each discrete point:

$$\int \mathcal{D}\phi(x) = \lim_{N \to \infty} \prod_{i=1}^{N} \int d\phi(x_i). \tag{1.6}$$

The action in Minkowski space for a scalar field with only nonderivative self-interactions and a canonical quadratic kinetic term is

$$\begin{aligned} S = \int d^4 x \mathcal{L} &= \int d^4 x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \right] \\ &= \int d^4 x \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} |\vec{\nabla} \phi|^2 - \frac{1}{2} m^2 \phi^2 - V(\phi) \right]. \end{aligned} \tag{1.7}$$

A generalization of the scalar field VEV is the *correlation function* or Green's function or n-point function,

$$G_n(x_1, \dots, x_n) = \langle 0 | T\{\hat{\phi}(x_1) \dots \hat{\phi}(x_n)\} | 0 \rangle = \int \mathcal{D}\phi e^{iS[\phi]} \phi(x_1) \dots \phi(x_n). \tag{1.8}$$

We note, however, that the weight inside the integral, e^{iS}, is highly oscillatory, so the n-point functions are hard to define precisely in Minkowski space.

It is much better to Wick rotate to Euclidean space, with signature $+ + \dots +$, define all objects there, and at the end Wick rotate back to Minkowski space. Both definitions and calculations are then easier. This is also what will happen in the case of AdS/CFT, which will have a natural definition in Euclidean signature, but will be harder to continue back to Minkowski space.

The Wick rotation happens through the relation $t = -it_E$. To rigorously define path integrals, we consider only paths which are periodic in Euclidean time t_E. In the case that the Euclidean time is periodic, a quantum mechanical path integral gives the statistical mechanics partition function at $\beta = 1/kT$ through the Feynman–Kac formula,

$$Z(\beta) = \mathrm{Tr}\{e^{-\beta\hat{H}}\} \left(= \int dq \sum_n |\psi_n(q)|^2 e^{-\beta E_n} = \int dq \langle q, \beta | q, 0\rangle \right)$$
$$= \int \mathcal{D}q e^{-S_E[q]}|_{q(t_E+\beta)=q(t_E)}. \tag{1.9}$$

To obtain the vacuum functional in quantum field theory, we consider the generalization to field theory, for periodic paths with infinite period, i.e. $\lim_{\beta\to\infty} \phi(\vec{x}, t_E + \beta) = \phi(\vec{x}, t_E)$. The Euclidean action is defined through Wick rotation, by the definition

$$iS_M \equiv -S_E. \tag{1.10}$$

This gives for (1.7)

$$S_E[\phi] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi)\right], \tag{1.11}$$

where, since we are in Euclidean space, $a_\mu b_\mu = a_\mu b^\mu = a_\mu b_\nu \delta^{\mu\nu}$, and time is defined as $t_M \equiv x^0 = -x_0 = -it_E$, $t_E = x_4 = x^4$, and so $x^4 = ix^0$. In this way, the oscillatory factor e^{iS} has been replaced by the highly damped factor e^{-S}, sharply peaked on the minimum of the Euclidean action.

The Euclidean space correlation functions are then defined as

$$G_n^{(E)}(x_1, \ldots, x_n) = \int \mathcal{D}\phi e^{-S_E[\phi]}\phi(x_1)\ldots\phi(x_n). \tag{1.12}$$

We can define a generating functional for the correlation functions, the *partition function*,

$$Z^{(E)}[J] = \int \mathcal{D}\phi e^{-S_E[\phi]+J\cdot\phi} \equiv {}_J\langle 0|0\rangle_J, \tag{1.13}$$

where in d dimensions

$$J \cdot \phi \equiv \int d^dx J(x)\phi(x). \tag{1.14}$$

It is so called because at finite periodicity β we have the same relation to statistical mechanics as in the quantum mechanical case,

$$Z^{(E)}[\beta, J] = \mathrm{Tr}\{e^{-\beta\hat{H}_J}\} = \int \mathcal{D}\phi e^{-S_E[\phi]+J\cdot\phi}|_{\phi(\vec{x},t_E+\beta)=\phi(\vec{x},t_E)}. \tag{1.15}$$

The Euclidean correlation functions are obtained from derivatives of the partition function,

$$G_n^{(E)}(x_1, \ldots, x_n) = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \int \mathcal{D}\phi e^{-S_E+J\cdot\phi}\bigg|_{J=0}$$
$$= \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z^{(E)}[J]\bigg|_{J=0}. \tag{1.16}$$

Going back to Minkowski space, we can also define a partition function as a generating functional of the Green's functions,

$$Z[J] = \int \mathcal{D}\phi e^{iS[\phi]+i\int d^dx J(x)\phi(x)}, \tag{1.17}$$

that again gives the correlation functions through its derivatives by

$$\begin{aligned} G_n(x_1,\dots,x_n) &= \frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)}\int \mathcal{D}\phi e^{iS+i\int d^dx J(x)\phi(x)}\Big|_{J=0} \\ &= \frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)} Z[J]\Big|_{J=0}. \end{aligned} \tag{1.18}$$

The correlation functions can be calculated in perturbation theory in the interaction S_{int}, through the use of Feynman diagrams.

The Feynman theorem relates the correlation functions in the full theory, in the vacuum of the full theory $|\Omega\rangle$, with normalized ratios of correlation functions in the interaction picture, in the vacuum of the free theory $|0\rangle$,

$$\begin{aligned} &\langle\Omega|T\{\phi_H(x_1)\dots\phi_H(x_n)\}|\Omega\rangle \\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_I(x_1)\dots\phi_I(x_n)\exp\left[-i\int_{-T}^{T} dtH_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T} dtH_I(t)\right]\right\}|0\rangle}, \end{aligned} \tag{1.19}$$

where H_I is the interaction Hamiltonian H_i in the interaction picture ($H = H_0 + H_i$), ϕ_I is an interaction picture field, and ϕ_H is a Heisenberg picture field. The denominator cancels vacuum bubbles, which factorize in the calculation, leaving only connected diagrams.

In the path integral formalism and in Euclidean space, we can find correlation functions of the full theory as normalized correlation functions in the interaction picture (divided by the vacuum bubbles), giving again connected diagrams only. For the one-point function and at nonzero source $J(x)$, we obtain the relation

$$\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\delta(-W[J])}{\delta J(x)}, \tag{1.20}$$

where $-W[J]$ is defined as the generating functional of connected diagrams, relation solved by

$$Z[J] = \mathcal{N}e^{-W[J]}. \tag{1.21}$$

Here $W[J]$ is called the free energy, again because of the relation with statistical mechanics.

To exemplify the Feynman rules, we use a scalar field action in Euclidean space,

$$S_E[\phi] = \int d^4x\left[\frac{1}{2}(\partial_\mu\phi)^2 + m^2\phi^2 + V(\phi)\right]. \tag{1.22}$$

Here I have used the notation

$$(\partial_\mu\phi)^2 = \partial_\mu\phi\partial^\mu\phi = \partial_\mu\phi\partial_\nu\phi\eta^{\mu\nu} = -\dot{\phi}^2 + (\vec{\nabla}\phi)^2. \tag{1.23}$$

Moreover, for concreteness, I use $V = \lambda\phi^4$.

Then, the **Feynman diagram in x space** is obtained as follows. One draws a diagram, in the example in Fig. 1.1a it is the so-called "setting Sun" diagram.

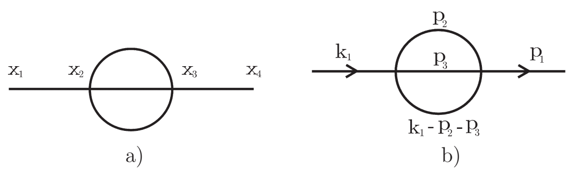

Figure 1.1 a) "Setting Sun" diagram in x-space; b) "Setting Sun" diagram in momentum space.

The Feynman rules are:

0. Draw all the Feynman diagrams for the given correlation function at the given loop order (or given number of vertices).

1. A line between point x and point y represents the Euclidean propagator

$$\Delta(x,y) = [-\partial_\mu\partial^\mu + m^2]^{-1} = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2+m^2}, \tag{1.24}$$

which is a Green's function for the kinetic operator, i.e.

$$[-\partial_\mu\partial^\mu + m^2]_x \Delta(x,y) = \delta(x-y). \tag{1.25}$$

The analytical continuation (Wick rotation) of the Euclidean propagator to Minkowski space gives the Feynman propagator,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2-i\epsilon}e^{ip\cdot(x-y)}. \tag{1.26}$$

2. A 4-vertex at point x represents the vertex factor

$$\int d^4x(-\lambda). \tag{1.27}$$

3. Then the value of the Feynman diagram, $F_D^{(N)}(x_1,\ldots,x_n)$ is obtained by multiplying all the above elements, and the value of the n-point function is obtained by summing over diagrams, and over the number of 4-vertices N with a weight factor $1/N!$:

$$G_n(x_1,\ldots,x_n) = \sum_{N\geq 0}\frac{1}{N!}\sum_{\text{diag }D} F_D^{(N)}(x_1,\ldots,x_n). \tag{1.28}$$

Equivalently, one can use a rescaled potential $\lambda\phi^4/4!$ and construct only *topologically inequivalent* diagrams. Then the vertices are still $\int d^4x(-\lambda)$, but we divide each inequivalent diagram by a statistical weight factor,

$$S = \frac{N!\,(4!)^N}{\#\text{ of equivalent diagrams}}, \tag{1.29}$$

which equals the number of symmetries of the diagram.

In momentum space, we can use simplified Feynman rules, where we consider as independent momenta the external momenta flowing into the diagram, and integration variables $l_1,\ldots,l_L$ for each independent loop in the diagram. Using momentum conservation at each vertex, we can calculate the momentum on each internal line, p_i, as a function of the loop momenta l_k and the external momenta q_j. The propagator is now

$$\Delta(p) = \frac{1}{p^2 + m^2}, \tag{1.30}$$

and for each internal line (between two internal points) we write $1/(p_i^2 + m^2)$, for each external line (between two points, one of them external) $q/(q_j^2 + m^2)$. The vertex factor is now simply $-\lambda$.

1.3 S matrices vs. correlation functions

We mentioned in the previous section that the VEV of a scalar field is an observable in quantum theory. More precisely, the normalized VEV in the presence of a source $J(x)$,

$$\begin{aligned}\phi(x;J) &= \frac{{}_J\langle 0|\hat{\phi}(x)|0\rangle_J}{{}_J\langle 0|0\rangle_J} = \frac{1}{Z[J]}\int \mathcal{D}\phi e^{-S_E[\phi]+J\cdot\phi}\phi(x) = \frac{\delta}{\delta J(x)}\ln Z[J] \\ &= -\frac{\delta}{\delta J(x)}W[J],\end{aligned} \tag{1.31}$$

is called the *classical field* $\phi^{\rm cl}$ and satisfies a quantum version of the classical field equation. One defines the *quantum effective action* as the Legendre transform of the free energy,

$$\Gamma[\phi^{\rm cl}] = W[J] + \int d^4x J(x)\phi^{\rm cl}(x), \tag{1.32}$$

and finds that it contains the classical action, plus quantum corrections. Then we have the quantum analog of the classical equation of motion with a source $\delta S[\phi]/\delta\phi(x) = J(x)$,

$$\frac{\delta\Gamma[\phi^{\rm cl}]}{\delta\phi^{\rm cl}(x)} = J(x). \tag{1.33}$$

The effective action is a generator of the one particle irreducible (1PI) diagrams (except for the 2-point function, where we add an extra term).

To relate to real scatterings, one constructs incoming and outgoing wavefunctions, representing actual states, in terms of the idealized states of fixed (external) momenta $\vec{k}$. These are Schrödinger picture states $\langle\{\vec{p}_i\}|$ and $|\{\vec{k}_j\}\rangle$. We also define Heisenberg picture states whose wavepackets are well isolated at $t = -\infty$, and can be considered noninteracting there (but overlap at other t),

$$|\{\vec{p}_i\} >_{\rm in}, \tag{1.34}$$

and Heisenberg picture states whose wavepackets are well isolated at $t = +\infty$, and can be considered noninteracting there (but overlap at other t),

$$|\{\vec{p}_i\} >_{\rm out} . \tag{1.35}$$

Then the S-matrix is defined by

$$\langle\{\vec{p}_i\}|S|\{\vec{k}_j\}\rangle = {}_{\rm out}\langle\{\vec{p}_i\}|\{\vec{k}_j\}\rangle_{\rm in}. \tag{1.36}$$

Reduction formula (Lehmann, Symanzik, Zimmermann)

The *LSZ formula* relates S-matrices to correlation functions in momentum space, in Minkowski space, near the physical pole for incoming and outgoing particles.

Define the momentum space Green's functions as

$$\tilde{G}_{n+m}(p_i^\mu, k_j^\mu) = \int \prod_{i=1}^{n} \int d^4 x_i e^{-ip_i\cdot x_i} \times \prod_{j=1}^{m} \int d^4 y_j e^{ik_j\cdot y_j} \langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\}|\Omega\rangle. \tag{1.37}$$

Then we have

$$\begin{aligned} &{}_{\rm in}\langle\{p_i\}_n|\{k_j\}_m\rangle_{\rm out} \\ &= \lim_{p_i^2\to -m_i^2, k_j^2\to -m_j^2} \prod_{i=1}^{n} \frac{(p_i^2+m^2-i\epsilon)}{-i\sqrt{Z}} \prod_{j=1}^{m} \frac{(k_j^2+m^2-i\epsilon)}{-i\sqrt{Z}} \tilde{G}_{n+m}(p_i^\mu, k_j^\mu). \end{aligned} \tag{1.38}$$

Here Z is the field renormalization factor, and can be defined from the behavior near the physical pole of the full 2-point function,

$$G_2(p) = \int d^4 x e^{-ip\cdot x}\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle \simeq \frac{-iZ}{p^2+m^2-i\epsilon}. \tag{1.39}$$

In other words, to find the S-matrix, we put the external lines on a shell, and divide by the full propagators corresponding to all the external lines (but note that Z belongs to two external lines, hence the $\sqrt{Z}$). This implies a diagrammatic procedure called *amputation*: we do not use propagators on the external lines. We also need to consider connected diagrams only, since the S-matrices are normalized objects, and we need to exclude processes where nothing happens and external particles go through without interactions, corresponding to the identity matrix. Therefore we have

$$\langle\{\vec{p}_i\}|S-1|\{\vec{k}_j\}\rangle = \left(\sum \text{connected, amputated Feynman diag.}\right) \times (\sqrt{Z})^{n+m}. \tag{1.40}$$

To understand the amputation procedure, consider the setting Sun diagram with external momenta k_1 and p_1 and internal momenta p_2, p_3 and $k_1 - p_2 - p_3$ in Fig. 1.1b. The result for the amputated diagram is in Euclidean space (note that for the S-matrix we must go to Minkowski space instead):

$$(2\pi)^4\delta^4(k_1-p_1)\int \frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\lambda^2\frac{1}{p_2^2+m^2}\frac{1}{p_3^2+m^2}\frac{1}{(k_1-p_2-p_3)^2+m_4^2}. \tag{1.41}$$

Feynman path integral with composite operators

Up to now we have considered only correlators of fundamental fields, which are related to external states for the quanta of these fields. But there is no reason to restrict ourselves to this, we can also consider external states corresponding to a composite field $\mathcal{O}(x)$, for instance

$$\mathcal{O}_{\mu\nu}(x) = (\partial_\mu\phi\partial_\nu\phi)(x)(+\cdots). \tag{1.42}$$

We can then define Euclidean space correlation functions for these operators

$$\begin{aligned} < \mathcal{O}(x_1)\cdots\mathcal{O}(x_n) >_{\text{Eucl}} &= \int \mathcal{D}\phi e^{-S_E}\mathcal{O}(x_1)\cdots\mathcal{O}(x_n) \\ &= \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\int \mathcal{D}\phi e^{-S_E+\int d^4x\mathcal{O}(x)J(x)}|_{J=0}, \end{aligned} \tag{1.43}$$

which can be obtained from the generating functional

$$Z_{\mathcal{O}}[J] = \int \mathcal{D}\phi e^{-S_E+\int d^4x\mathcal{O}(x)J(x)}. \tag{1.44}$$

1.4 Electromagnetism, Yang–Mills fields and gauge groups

Electromagnetism

Up to now we have discussed only scalar fields. Gauge bosons describing forces between particles correspond to vector fields. The simplest example of such a field is the Maxwell field describing the electromagnetic force (the photon),

$$A_\mu(x) = (-\phi(\vec{x},t), \vec{A}(\vec{x},t)), \tag{1.45}$$

where ϕ is the Coulomb potential and $\vec{A}$ is the vector potential.

The field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv 2\partial_{[\mu}A_{\nu]}, \tag{1.46}$$

and it contains the electric $\vec{E}$ and magnetic $\vec{B}$ fields as

$$-F_{0i} = F^{0i} = E^i; \quad F_{ij} = \epsilon_{ijk}B^k. \tag{1.47}$$

The observables like $\vec{E}$ and $\vec{B}$ are defined in terms of $F_{\mu\nu}$ (and not A_μ) and as such the theory has a gauge symmetry under a $U(1)$ group, that leaves $F_{\mu\nu}$ invariant,

$$\delta A_\mu = \partial_\mu\lambda; \quad \delta F_{\mu\nu} = 2\partial_{[\mu}\partial_{\nu]}\lambda = 0. \tag{1.48}$$

The Minkowski space action for electromagnetism is

$$S_{\text{Mink}} = -\frac{1}{4}\int d^4xF_{\mu\nu}^2, \tag{1.49}$$

which becomes in Euclidean space (since A_0 and $\partial/\partial x^0 = \partial_t$ rotate in the same way)

$$S_E = \frac{1}{4}\int d^4x(F_{\mu\nu})^2 = \frac{1}{4}\int d^4xF_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}. \tag{1.50}$$