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1 Scalar-Wave Models in Electromagnetic Scattering

This chapter is concerned with the scalar forward models used in microwave (MW) imaging. These are mathematical models of varying degrees of accuracy that predict the field based on a known source of radiation in a known environment. They are called *forward* because they describe the causal (or forward-in-time) relationship in a phenomenon we could express as *cause* → *effect*. In imaging, the *cause* is described by the *model parameters*, i.e., (i) the parameters of the sources generating the field and (ii) the parameters of the environment where this field exists or propagates.[1] The *effect* is described by the *observation data*, or simply, the *data*. These are signals acquired through measurements. Thus, in imaging, the forward model predicts the data, provided the model parameters are known.

The object of imaging, however, is the *inverse problem*, which, in contrast to the forward problem, is expressed as *effect* → *cause*. Finding what caused an effect is not an easy task. The second part of this book is dedicated to the mathematical methods used to accomplish this task. For now, it suffices to say that we first need to have a forward model of a phenomenon before we can start solving inverse problems based on this phenomenon. To illustrate this point, imagine that you are listening to a recording of a symphony; in order to tell which instruments play at any given time, you have first to have heard the sound of each instrument.

The phenomenon of interest in the forward models of MW imaging is the scattering of the high-frequency electromagnetic (EM) field by objects.[2] The scattering objects are often referred to as *targets*, especially in radar, or as *scatterers*. In this chapter, we discuss the mathematical scalar models of scattering.

The EM field is a vectorial field fundamentally described by Maxwell's equations [1, 2, 3, 4, 5]. For a summary of Maxwell's equations, see Appendix A. However, to simplify the analysis, scalar approximations are often made, and here we start with these simpler models. The scalar-wave model is very useful as an intermediate step toward the understanding of the vectorial wave model. It can also serve as a bridge to understanding acoustic and elastic wave phenomena, which are widely used in imaging.

Strictly speaking, the scalar-wave model in electromagnetism is limited to the case of a uniform isotropic medium, which becomes apparent when one attempts to reduce

[1] The term *propagation* refers to the manner in which a field develops in space and time.
[2] The *scattering* of waves refers to the way the original, or *incident*, waves interact with obstacles. This interaction produces secondary, or *scattered*, waves that often spread away from the object in various directions; thus, the term *scatter*.

Maxwell's equations to decoupled scalar second-order partial differential equations [1, 3, 5]. In such a medium, as discussed later, we may work with two types of scalar functions: (i) the *Cartesian* components of the electric and magnetic field vectors and (ii) the so-called *wave functions*.

One may wonder why we are interested in the simple scenario of uniform isotropic medium, bearing in mind that MW imaging is inherently concerned with nonuniform objects. The short answer is that most MW imaging methodologies assume that the object under test (OUT) is immersed in a uniform medium. And in most applications, this uniform medium is predominantly isotropic; for example, air, concrete, sea water, soil, etc. Even if the assumption of uniformity is invalid, which may be the case when we deal with imaging in complex environments (e.g., living tissue or concealed weapon detection), it helps to first understand how imaging is done in a uniform background and then move on to complex environments.

1.1 Partial Differential Equations for Scalar Waves in the Time Domain

Before exploring in depth the mathematics of scalar waves, the reader should be aware of an important physical limitation of the analytical time-domain models of electromagnetism discussed later: they are applicable only if the frequency dependence (or the dispersion) of the medium properties is negligible. This, of course, cannot be true throughout the spectrum; however, it could be approximately true for the bandwidth of the radiation (the bandwidth of the excitation sources). Then, these models can be useful. Time-domain modeling is particularly important in imaging with pulsed radar. More notes on dispersion are given later as appropriate.

The time-domain Maxwell's equations, when applied to the *Cartesian* components of the field vectors in a uniform isotropic medium, lead to the second-order partial differential equation [3, 5] (see Appendix B for its derivation):

$$\left[\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2} - (\mu\sigma_{\mathrm{e}} + \epsilon\sigma_{\mathrm{m}})\frac{\partial}{\partial t} - \sigma_{\mathrm{e}}\sigma_{\mathrm{m}}\right] u_\xi(\mathbf{r}, t) = \mathcal{S}_\xi(\mathbf{r}, t), \tag{1.1}$$

where $\mathbf{r}$ is position, t is time, $u \equiv \mathcal{E}$ or $\mathcal{H}$ ($\mathcal{E}$ or $\mathcal{H}$ denote the type of field, electric or magnetic, respectively), and $\xi \equiv x, y, z$ is the vector component. The constitutive parameters ϵ, μ, σ_{e}, and σ_{m} are permittivity, permeability, electric conductivity, and magnetic conductivity, respectively,[3] all of them being constant in $\mathbf{r}$ as per the assumption of a

[3] In high-frequency problems, the conductivities σ_{e} and σ_{m} are often referred to as *equivalent* conductivities, and they represent losses due to the conversion of EM energy into heat (dissipation). The equivalent magnetic conductivity σ_{m} is zero at DC (direct, or steady, current), reflecting the fact that magnetic charges and, therefore, magnetic conduction currents do not exist. At higher frequencies, however, magnetic materials do exhibit polarization loss analogous to the one observed in polarizable dielectrics. The difference with lossy dielectrics is that, in the latter, loss due to charge transport (electric conduction) is present and this loss mechanism exists all the way down to DC.

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uniform medium. The excitation term is

$$\mathcal{S}_\xi(\mathbf{r},t) = \begin{cases} \left(\mu\frac{\partial}{\partial t} + \sigma_\mathrm{m}\right)\mathcal{J}_{\mathrm{i},\xi}(\mathbf{r},t) + \left(\epsilon\frac{\partial}{\partial t} + \sigma_\mathrm{e}\right)^{-1}\frac{\partial\rho_\mathrm{i}}{\partial\xi} & \text{if } u = \mathcal{E} \\ -(\nabla\times\boldsymbol{\mathcal{J}}_\mathrm{i})_\xi & \text{if } u = \mathcal{H}, \end{cases} \tag{1.2}$$

where ρ_i and $\boldsymbol{\mathcal{J}}_\mathrm{i}$ are the *impressed* electric charge and current densities, respectively. As discussed in Appendix B, $(\epsilon\frac{\partial}{\partial t} + \sigma_\mathrm{e})^{-1}$ denotes the inverse of the differential operator $(\epsilon\frac{\partial}{\partial t} + \sigma_\mathrm{e})$.

Eq. (1.1) is a linear second-order partial differential equation describing damped waves, and it appears in analogous forms in various physical fields; see, for example, [6, 7, 8].

It is important to note that Eq. (1.1) follows from Maxwell's curl equations only after imposing the conditions (see Appendix B)

$$\nabla\cdot\boldsymbol{\mathcal{B}}(\mathbf{r},t) = 0 \tag{1.3}$$

$$\nabla\cdot\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \rho_\mathrm{e}(\mathbf{r},t), \tag{1.4}$$

where $\boldsymbol{\mathcal{D}}$ is the electric flux density, $\boldsymbol{\mathcal{B}}$ is the magnetic flux density, and ρ_e is the *total* electric charge density. Eq. (1.3) and Eq. (1.4) are Maxwell's divergence equations. For a summary of Maxwell's equations, see Appendix A.

The solutions to Eq. (1.1) satisfy Maxwell's curl equations if and only if they also satisfy Eq. (1.3–1.4) [1]. In other words, *for a solution to Eq. (1.1) to be admissible, it must be checked against Eq. (1.3–1.4).*

Another limitation of Eq. (1.1) is that it holds for the *Cartesian* components of the field only. This is inconvenient when the field has spherical or cylindrical symmetries. This limitation is overcome by the use of the *wave functions*. The wave functions are in effect the values of two collinear vector potentials, the magnetic vector potential $\boldsymbol{\mathcal{A}}$ and the electric vector potential $\boldsymbol{\mathcal{F}}$, which are so defined as to have a fixed known direction $\hat{\mathbf{u}}$, i.e., $\boldsymbol{\mathcal{A}} = \hat{\mathbf{u}}\mathcal{A}$, $\boldsymbol{\mathcal{F}} = \hat{\mathbf{u}}\mathcal{F}$. The wave functions are $\mathcal{A}$ and $\mathcal{F}$. The construction of EM solutions in a uniform source-free medium using Cartesian, cylindrical, and spherical scalar-wave functions is described in detail in [1, 5]. Also, Appendix C summarizes the methods used to reduce the EM model to two decoupled scalar-wave equations. What matters here is that the two wave functions satisfy the wave equation (see Eq. (C.20) in Appendix C):

$$\left[\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2} - (\mu\sigma_\mathrm{e} + \epsilon\sigma_\mathrm{m})\frac{\partial}{\partial t} - \sigma_\mathrm{e}\sigma_\mathrm{m}\right]u(\mathbf{r},t) = 0, \tag{1.5}$$

where $u(\mathbf{r},t) \equiv \mathcal{A}, \mathcal{F}$ is the wave function.

It must be emphasized that, with the proper choice of the wave functions and their polarization $\hat{\mathbf{u}}$, the above equation Eq. (1.5) can be used in rectangular, cylindrical, or spherical coordinate systems. Moreover, unlike Eq. (1.1), the wave-function model needs to solve at the most two (not three) decoupled scalar equations. Often, one equation suffices, for example for a transverse wave in a uniform source-free medium. Thus the main advantages of the wave-function models are that (i) they can be employed

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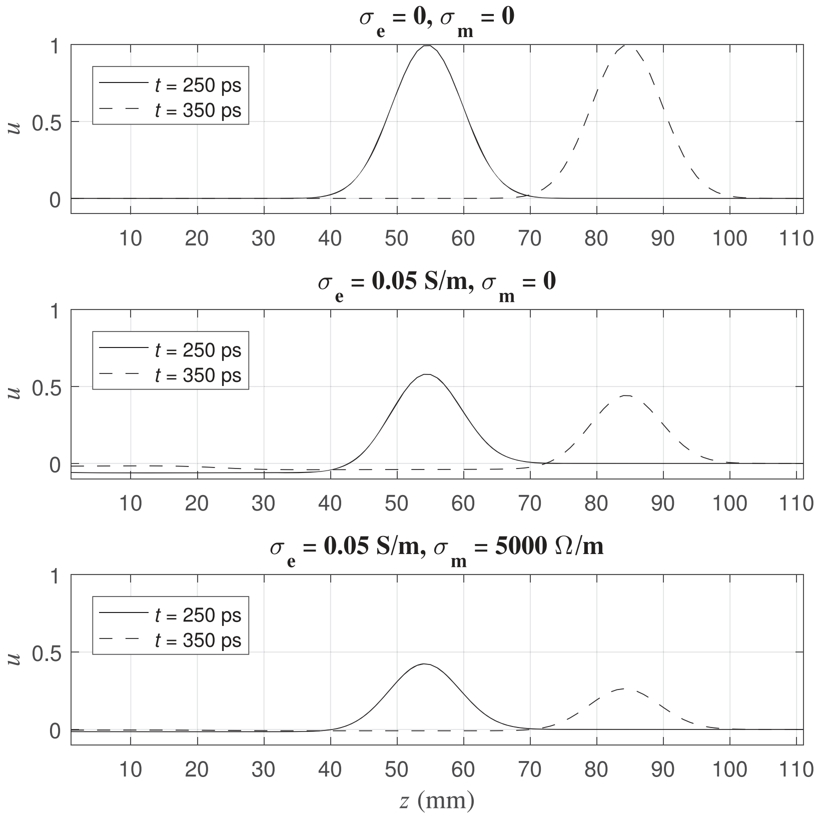

Figure 1.1 Illustration of the impact of the electric and magnetic conductivities on the attenuation of waves: the spatial distribution of a uniform plane wave simulated with MEFiSTo-3D [10].

not only in rectangular but also in curvilinear coordinate systems and (ii) at the most two decoupled scalar equations need to be solved as opposed to three such equations when solving for the vector field in Cartesian coordinates. Notice also that the left side of Eq. (1.5) contains the same partial differential operator (knowns as the *wave* or *d'Alambert* operator) as that in Eq. (1.1).

Let me emphasize that in general the constitutive parameters in Eq. (1.1) and Eq. (1.5) exhibit frequency dependence [9]. This is why Eq. (1.1) and Eq. (1.5) are used only if the excitation signals span a frequency range within which the constitutive parameters are sufficiently constant. The most obvious reminder of the frequency dispersion is the fact that at static and quasi-static (very low frequency) regimes, $\sigma_m = 0$. In such regimes, the time-derivative terms in Eq. (1.1) and Eq. (1.5) are set to zero, reducing these equations to the Laplace form $\nabla^2 u = 0$, *not* to $(\nabla^2 - \sigma_e \sigma_m)u = 0$.

To illustrate the impact of the conductivity terms σ_e and σ_m in the wave equation Eq. (1.5), let us consider a uniform plane wave propagating along z. Fig. 1.1 shows the wave as a function of position in two different instances, $t = 250$ ps and $t = 350$ ps, in three cases: (a) when the medium (vacuum, $\epsilon = \epsilon_0$, $\mu = \mu_0$) has no loss ($\sigma_e = 0$,

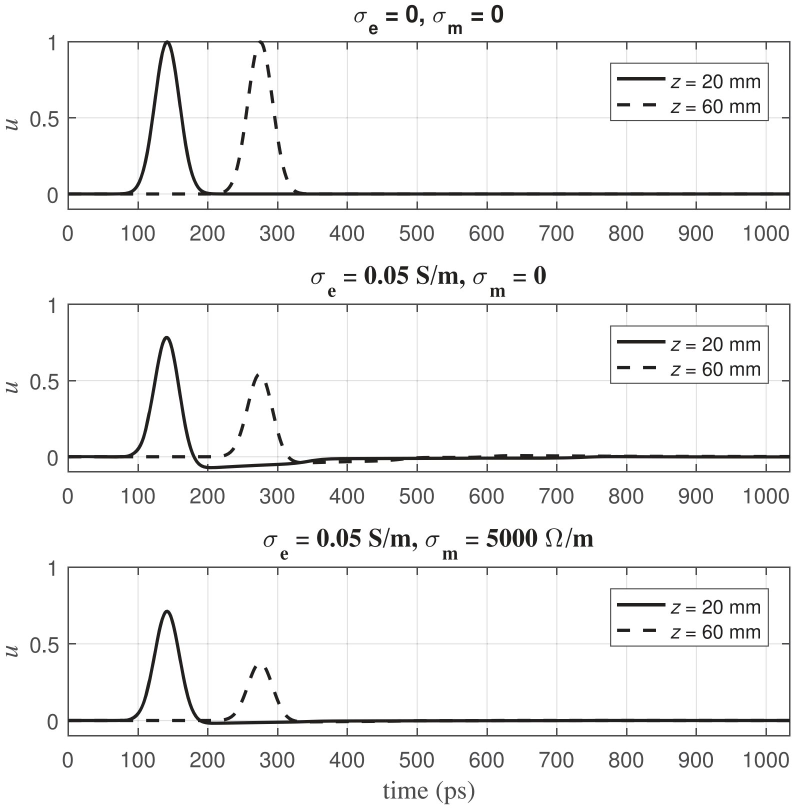

Figure 1.2 Illustration of the impact of the electric and magnetic conductivities on the attenuation of waves: the temporal distribution of a uniform plane wave simulated with MEFiSTo-3D [10].

$\sigma_m = 0$); (b) when the medium has only loss due to electric conductivity ($\sigma_e = 0.05$ S/m, $\sigma_m = 0$); and (c) when the medium has both electric and magnetic loss ($\sigma_e = 0.05$ S/m, $\sigma_m = 5000$ Ω/m). Fig. 1.2 shows the same wave as a function of time measured at two locations: $z = 20$ mm (closer to the source) and $z = 60$ mm. It is clear that losses are responsible for a decrease in strength as the wave propagates away from the source. This decrease is referred to as *dissipation* (the conversion of EM energy into heat). Magnetic and electric specific conductivities are both mathematical means of describing dissipation. Loss also causes changes in the shape of the signal. This is referred to as *dispersion*. Notice the tails behind the main pulse that are clearly visible in the case of losses in both Fig. 1.1 and Fig. 1.2.

1.2 Plane, Spherical, and Cylindrical Waves in the Time Domain

Let us consider the solutions of the wave equations in cases where the wave is independent in two of the three spatial coordinates. They give the mathematical form of the plane, spherical, and cylindrical waves depending on the chosen coordinate system. These solutions are important because they are often used to approximate the field due to distant RF or MW sources. In imaging, these are approximations of what is referred to as the *incident field*, i.e., the field that exists in the background medium when no scattering objects are present. The time-dependent plane and spherical-wave solutions are widely used in pulsed radar imaging.

For simplicity, let us focus on the time-domain symmetric wave solutions in the loss-free case ($\sigma_e = 0$ and $\sigma_m = 0$) when the wave equation is

$$\left(\nabla^2 - \frac{1}{\upsilon^2}\frac{\partial^2}{\partial t^2}\right) u(\mathbf{r}, t) = s(\mathbf{r}, t). \tag{1.6}$$

Here, υ is the speed at which the wavefront advances, and $\upsilon^{-2} = \mu\epsilon$. This case has practical significance. It is true that matter always exhibits loss (or dissipation), however minuscule it may be. Yet, for some forms of matter (e.g., air, most ceramics), the assumption of no loss holds very well at MW frequencies. In addition, the general solutions to Eq. (1.5), where the loss terms are nonzero, are not available in a closed analytical form, unless some assumptions are made; for example, the assumption of low loss [7, 8]. In any case, due to the significant frequency dependence of the damping rates, the analytical modeling of a lossy medium is best done in the frequency domain, which we pursue in the subsequent sections.

A. Plane-Wave Solution in the Time Domain

If the wave field is independent of two Cartesian spatial variables, it is referred to as a uniform plane wave, described by what is known as the equation of the vibrating string. For example, if the wave is independent of x and y, then the source-free (or homogeneous) form of Eq. (1.6) is

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{\upsilon^2}\frac{\partial^2}{\partial t^2}\right) u(z, t) = 0. \tag{1.7}$$

The general solution of Eq. (1.7) is [11]

$$u(z, t) = f_+(z - \upsilon t) + f_-(z + \upsilon t). \tag{1.8}$$

Here, f_+ and f_- are arbitrary differentiable functions representing waves propagating in the positive and in the negative z directions, respectively. It is common to refer to the first term, the argument of which is $p^+ = z - \upsilon t$, as the *incident* wave, while the second term, of argument $p^- = z + \upsilon t$, is the *reflected* wave.[4]

[4] The reader is reminded that the particular form of f_+ and f_- is determined once the initial or the boundary conditions are given. For example, if nonzero initial conditions $u(z, 0) = u_0(z)$ and $\frac{\partial u(z,0)}{\partial t} = g_0(z)$ are imposed, then [11]

$$u(z, t) = \tfrac{1}{2}\left[u_0(z - \upsilon t) + u_0(z + \upsilon t) + \tfrac{1}{\upsilon}\int_{x-\upsilon t}^{x+\upsilon t} g_0(\xi)d\xi\right].$$

Here, $u_0(\xi)$ and $g_0(\xi)$ are known functions. In a different scenario, the initial conditions may be zero, but a solution may be required for $z \geq 0$ such that $u(0, t) = b(t)$ where $b(t)$ is a known time-varying boundary condition at $z = 0$. Then $u(z, t) = b(t - z/\upsilon)$, $0 \leq z < \infty$.

Electrical engineers will recognize this one-dimensional (1D) wave behavior as the solution to the telegrapher's equation describing the propagation of voltage and current signals along a loss-free transmission line [12]. Eq. (1.8) also represents a uniform plane wave.

Another important class of EM waves, the *transverse electromagnetic (TEM)* waves,[5] also exhibit this behavior along z. In contrast to the uniform plane wave, the TEM wave may vary along the transverse coordinates x and y (as a harmonic function). More specifically, a TEM field component F_ξ, where ξ can be either x or y but *not* z, has the form $F_\xi = \ell(x,y) \cdot f(z \mp vt)$ (the sign depends on whether the wave is incident or reflected), where $\ell(x,y)$ must satisfy the 2D Laplace equation in the xy plane [1, 13]. Notice that the uniform plane wave is a particular case of the TEM wave for which $\ell(x,y) = const$.

The solution of the 3D wave equation Eq. (1.6) for a uniform plane wave propagating in any direction given by the unit vector $\hat{\mathbf{u}}$ is known as the general *one-way wave solution* [11]:

$$u(t,\mathbf{r}) = f(\hat{\mathbf{u}} \cdot \mathbf{r} - vt), \tag{1.9}$$

where $\mathbf{r} = (x,y,z)$ denotes the observation location. The first term in Eq. (1.8) is a special case of Eq. (1.9) when $\hat{\mathbf{u}} = \hat{\mathbf{z}}$, while the second term corresponds to the case when $\hat{\mathbf{u}} = -\hat{\mathbf{z}}$.

The plane-wave solution Eq. (1.9) is widely used in radar imaging to approximate the EM field in a region, which is (a) in open space and far from the sources and (b) sufficiently small to ignore the typical far-zone ($\sim 1/r$) behavior of the field intensity (r being the distance to the source).[6]

B. Spherical-Wave Solution in the Time Domain

Another symmetric solution of Eq. (1.6) arises when the field propagates as a spherical wave, i.e., it is independent of the two angular coordinates θ and ϕ of a spherical coordinate system. It is a function of the radial distance r and the time t. This solution plays an important role in the modeling of 3D open problems[7] where the spherical coordinate system is convenient to use. The usual definition of a spherical coordinate system is illustrated in Fig. 1.3. In this case, the Laplacian operator in Eq. (1.6) is written out in spherical coordinates, and the derivatives with respect to

[5] The term *transverse electromagnetic* (TEM) indicates that both the electric and the magnetic field vectors are perpendicular to the direction of propagation.

[6] The far zone of a radiating structure (an antenna) is all space beyond a distance r_f from the antenna such that it satisfies all of the following conditions: $r_f \gg \lambda$, $r_f \gg D_A$, and $r_f \geq 2D_A^2/\lambda$, where λ is the wavelength of the radiation and D_A is the maximum dimension of the antenna. In practice, the $\gg$ inequality above is usually taken as $> 10\times$. We recall that the far-zone EM field in an unbounded medium is a TEM field i.e., both the $\boldsymbol{\mathcal{E}}$ and the $\boldsymbol{\mathcal{H}}$ field vectors are transverse to $\hat{\mathbf{u}}$ [5]. It is these transverse field components that are approximated by Eq. (1.9). The transverse nature of the far-zone EM wave, together with the assumption that its dependence on the transverse coordinates can be neglected, ensures that the uniform plane-wave solution satisfies not only the wave equation but also Eq. (1.3–1.4).

[7] An *open* (or *radiation*) problem is that of analyzing the field in an infinite (or unbounded) region, the boundary of which extends to infinity.

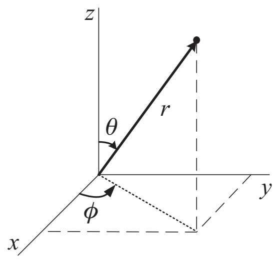

Figure 1.3 The spherical coordinate system.

θ and ϕ are set to zero. For the case of zero sources, the result is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)-\frac{1}{\upsilon^2}\frac{\partial^2 u}{\partial t^2}=0. \tag{1.10}$$

The general solution to this homogeneous equation is easily found since it can be reduced to the equation of the vibrating string Eq. (1.7) by rewriting it as [14]

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{\upsilon^2}\frac{\partial^2}{\partial t^2}\right)(ru)=0. \tag{1.11}$$

The general spherical-wave solution is then

$$u(r,t)=\frac{f_+(r-\upsilon t)}{r}+\frac{f_-(r+\upsilon t)}{r},\quad \xi=x,y,z. \tag{1.12}$$

The argument $p^+=r-\upsilon t$ in the first term of Eq. (1.12) implies a wave diverging from the center of the coordinate system, i.e., propagating in the positive radial direction $\hat{\mathbf{r}}$. This case corresponds to the outgoing wave of a point source of waveform $f_+(t)$ located at the origin (0, 0, 0). Thus, the first term in Eq. (1.12) is a physically valid *causal* solution to the open problem with a point source at the origin.[8] In contrast, $p^-=r+\upsilon t$ implies a spherical wave converging toward the origin with time. It could be understood as a spherical wave propagating backward (collapsing) toward the point source. This solution is not causal in an unbounded medium and is normally excluded from the forward model. Note, however, that a region bounded by a spherical reflecting surface (for example, a metallic shell) can support perfectly well the solution of the collapsing wave. In this case, the source of the collapsing wave is the spherical shell boundary, and the collapsing wave is in fact causal.

It is instructive to examine the validity of the scalar spherical-wave approximation of the EM waves in the form of Eq. (1.12) with regard to the conditions in Eq. (1.3–1.4). First, we note that the spherical components of the EM field do not satisfy the wave equation Eq. (1.1): their Cartesian components do. Therefore, Eq. (1.12) is not applicable to the wave Cartesian components in general.

[8] A causal solution describes a response (such as a field value) that never precedes its excitation source. For example, if the source function is identically zero everywhere in space before some initial time t_0, then the causal field solution must be identically zero everywhere in space for all $t \le t_0$.

The spherical-wave functions $\mathcal{A}$ and $\mathcal{F}$, however, do satisfy the wave equation in spherical coordinates (see Appendix C). As a consequence, it can be shown that all the components of the EM field that observe the $\sim 1/r$ behavior (these are referred to as the *far-zone* or *far-field* components) are transverse to the radial direction, i.e., the direction of propagation [5]. Thus, the far-zone wave is a TEM wave.[9] This result is often used in antenna engineering [15]. In particular, in a spherical coordinate system, the far field has only θ and ϕ components: u_ξ, $u \equiv \mathcal{E}, \mathcal{H}$, $\xi = \theta, \phi$, both of which, in general, are functions of θ and ϕ[10] in addition to their dependence on the distance as $\sim 1/r$. These field components are often approximated in the form of the first (outgoing) term in Eq. (1.12). Such a field, however, does violate Maxwell's divergence equations, Eq. (1.3–1.4).

Take as an example the far-zone electric field due to a current element[11] oriented along z and centered at the origin. Its far-zone $\boldsymbol{\mathcal{E}}$-field has a θ component only, and it behaves as $\mathcal{E}_\theta \sim \sin(\theta)/r$. The divergence of this $\boldsymbol{\mathcal{E}}$-field, found to be $\sim \cos(\theta)/r^2$, is not zero for all θ, thus violating Eq. (1.4). If this result is viewed as an error in the field approximation, we can state that this error decreases with distance as $\sim 1/r^2$ and is zero at $\theta = 90^\circ$. Notice that the plane $\theta = 90^\circ$ (the $z = 0$ plane) is a plane of symmetry in the field pattern where the radiation attains its maximum. This result can be generalized for any antenna by viewing it as a collection of current elements. *The spherical-wave approximation of any of the transverse field components is valid for any observation direction as long as r is sufficiently large. Its accuracy improves for directions close to the direction of maximum radiation* (the maximum of the radiation pattern) where the derivatives of the far field with respect to the angles θ and ϕ are zero.

C. Cylindrical-Wave Solution in the Time Domain

Analogously to the plane and spherical waves, the cylindrical wave is described by the solution of the wave equation in cylindrical coordinates, where the wave is set to be independent of two coordinates: the vertical coordinate z and the angular coordinate ϕ. The usual definition of the cylindrical coordinate system is illustrated in Fig. 1.4. The wavefront of the cylindrical wave is an infinitely long (along z) cylindrical surface that advances radially away from the z axis along the positive radial direction $\hat{\boldsymbol{\rho}}$. Clearly, such an infinite wavefront is an approximation that can be only locally valid. This approximation can be useful in the two-dimensional (2D)

[9] Strictly speaking, the far-zone EM field is only *predominantly* TEM because the longitudinal field components (either $\mathcal{E}_r$ or $\mathcal{H}_r$ or both) are only asymptotically zero and therefore negligible in comparison with the transverse components.

[10] The far-field dependence on the observation direction (θ, ϕ) is described by what is known as the *antenna radiation pattern*.

[11] A current element, also referred to as an infinitesimal electric dipole, is a vanishingly short line segment supporting alternating current (AC). The practical implementation of a current element is a very short (compared to the wavelength λ) piece of wire excited by an RF source at its midpoint. The current element, measured in units of (A $\times$ m), is the fundamental point source in radiation theory analogous to the point charge in electrostatics.

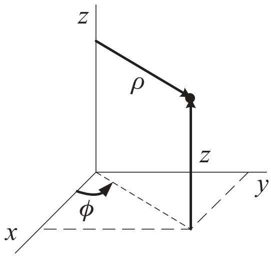

Figure 1.4 The cylindrical coordinate system.

time-domain modeling of open problems when the field and its sources can be assumed to be independent of the vertical variable z.

In this case, the Laplacian operator ∇^2 in Eq. (1.6) is written out in cylindrical coordinates, and the partial derivatives with respect to ϕ and z are set to zero. The result is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) - \frac{1}{\upsilon^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{1.13}$$

As it turns out, the general wave solution in this case is not as simple as in the cases of planar and spherical symmetry; it appears in the form of convolution integrals [16]:

$$u(\rho,t) = \int_{-\infty}^{t-\rho/\upsilon} \frac{f_+(\tau)d\tau}{\sqrt{(t-\tau)^2-(\rho/\upsilon)^2}} + \int_{t+\rho/\upsilon}^{\infty} \frac{f_-(\tau)d\tau}{\sqrt{(\tau-t)^2-(\rho/\upsilon)^2}}. \tag{1.14}$$

That the first term represents an outgoing wave becomes apparent from the fact that it is the *past* values of f_+ that contribute to $u(\rho,t)$. Note that ρ/υ is real-positive and that the upper limit of the integral ensures that the integrand remains real-valued for all τ. In contrast, in the second term, the integration is over the *future* values of f_-. In problems unbounded in ρ ($0 \le \rho < \infty$), the second term is nonphysical because it is acausal.

To understand better the behavior of the cylindrical wave, let us consider a particular outgoing wave solution when $f_+(t) = \delta(t)$ and $f_-(t) = 0$. Then, as per the sampling property of the δ-function, we have

$$u(\rho,t) = \begin{cases} 0 & \text{if } 0 < t < \rho/\upsilon \\ \frac{1}{\sqrt{t^2-(\rho/\upsilon)^2}} & \text{if } t > \rho/\upsilon. \end{cases} \tag{1.15}$$

As we will see shortly, this is in fact a solution proportional to Green's function of Eq. (1.13) [17]. This solution can be interpreted as an outward propagating wave behaving as $\sim 1/\sqrt{\rho}$. This becomes apparent if $u(\rho,t)$ is expressed as

$$u(\rho,t) = \begin{cases} 0 & \text{if } 0 < t < \rho/\upsilon \\ \frac{1}{\sqrt{t-\rho/\upsilon}} \cdot \frac{1}{\sqrt{t+\rho/\upsilon}} & \text{if } t > \rho/\upsilon, \end{cases} \tag{1.16}$$