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Introduction

The topic of this book is the application of mathematics to physical problems. Mathematics and physics are often taught separately. Despite the fact that education in physics relies on mathematics, many students consider mathematics to be disjointed from physics. Although this point of view may strictly be correct, it becomes problematic when it concerns an education in the sciences. The reason for this is that mathematics is the *only* language at our disposal for quantifying physical processes. Furthermore, common mathematical foundations often form the link between different topics in physics.

A language cannot be learned by just studying a textbook. In order to truly learn how to use a language, one has to go abroad and start using that language. By the same token, one cannot learn how to use mathematics in the physical sciences by just studying textbooks or attending lectures; the only way to achieve this is to venture into the unknown and apply mathematics to physical problems. The goal of this book is to do exactly that; we present exercises to apply mathematical techniques and knowledge to physical concepts. These are not presented as well-developed theory, but as a number of problems that elucidate the issues at stake. In this sense, the book offers a guided tour: material for learning is presented, but true learning will only take place by active exploration. The interplay of mathematics and physics is essential in the process; mathematics is the natural language to describe physics, while physical insight allows for a better understanding of the mathematics that is presented.

How can you use this book most efficiently?

Since this book is written as a set of problems, you may frequently want to consult other material as well to refresh or deepen your understanding of material. In many places we refer to the book by Boas (2006). In addition, the books by Butkov (1968), Riley et al. (2006), and Arfken and Weber (2005) on mathematical physics are excellent.

In addition to books, colleagues in either the same field or other fields can be a great source of knowledge and understanding. Therefore, do not hesitate to work together with others on these problems if you are in the fortunate position to do so. This may not only make the work more enjoyable, it may also help you in getting "unstuck" at difficult moments and the different viewpoints of others may help to deepen yours.

For whom is this book written?

This book is set up with the goal of obtaining a good working knowledge of mathematical physics that is needed for students in physics or geophysics. A certain basic knowledge of calculus and linear algebra is required to digest the material presented here. For this reason, this book is meant for upper-level undergraduate students or lower-level graduate students, depending on the background and skill of the student. In addition, teachers can use this book as a source of examples and illustrations to enrich their courses.

This book is evolving

This book will be improved regularly by adding new material, correcting errors, and making the text clearer. The feedback of both teachers and students who use this material is vital in improving this text, please send your remarks to

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Errata can be found at the following website: `http://www.mines.edu/~rsnieder/Errata.html`

Preface to the third edition

The third edition is a thorough revision and expansion of previous editions. We have added material throughout the text to help strengthen the learning process and provide essential material, and we have modified the problems so that the level of difficulty between problems is more balanced. We have added new chapters that cover Probability and Statistics (Chapter 21) and Inverse Problems (Chapter 22).

Acknowledgments

This book resulted from courses taught at Utrecht University, the Colorado School of Mines, Boise State University, and the University of Auckland. The remarks, corrections, and the encouragement of a large number of students have been very important in its development. It is impossible to thank all the students, but we especially want to thank the feedback from Jojanneke van den Berg, Jehudi Blom, Sterre Dortland, Thomas Geenen, Wiebe van Driel, Luuk van Gerven, Aaron Girard, Nathan Hancock, Noor Hogeweg, Wouter Kimman, and Frederiek Siegenbeek. In their role as teaching assistants, Dirk Kraaipoel and Jesper Spetzler have helped greatly in improving this book. Huub Douma has spent numerous hours at sea correcting earlier drafts. A number of colleagues have helped us very much with their comments; we especially want to mention Jami Johnson, Philip Cheng, Freeman Gilbert, Minhhuy Hô, Ted Jacobson, Alexander Kaufman, Antoine Khater, Rajmohan Kombiyil, Ken Larner, Sum Mak, Thijs Ruigrok, Hidetosi Sirai, Alex Summer, Jeannot Trampert, and John Wettlaufer.

We also want to acknowledge all the people who contributed to the open-source software used to help create this book. Typesetting was done in LaTeX, and python was used for the computations (including obspy for the seismology applications). Many of the figures were first drafted by Barbara McLenon in Xfig, who we thank for her support, but most of the figures in the 3rd edition were made in Inkscape.

The support and advice of Matt Lloyd, Adam Black, Eoin O'Sullivan, Jayne Aldhouse, Maureen Storey, Simon Capelin, and Philip Alexander of Cambridge University Press has been very helpful and stimulating during the preparation of this work. Lastly, we want to thank everybody who helped us in numerous ways to make writing this book such a joy.

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Dimensional analysis

The material of this chapter is usually not covered in a book on mathematics. The field of mathematics deals with numbers and numerical relationships. It does not matter what these numbers are; they may account for physical properties of a system, but they may equally well be numbers that are not related to anything physical. Consider the expression $g = df/dt$. From a mathematical point of view these functions can be anything, as long as g is the derivative of f, with respect to t. The situation is different in physics. When $f(t)$ defines the position of a particle, and t denotes time, then $g(t)$ is a velocity. This relation fixes the physical dimension of $g(t)$. In mathematical physics, the physical dimension of variables imposes constraints on the relation between these variables. In this chapter we explore these constraints. In Section 2.2 we show that this provides a powerful technique for spotting errors in equations. In the remainder of this chapter we show how the physical dimensions of the variables that govern a problem can be used to find physical laws. Surprisingly, while most engineers learn about dimensional analysis, this topic is not covered explicitly in many science curricula.

2.1 Two rules for physical dimensions

In physics every physical parameter is associated with a physical dimension. The value of each parameter is measured with a certain physical unit. For example, when we measure how long a table is, the result of this measurement has dimension "length." This length is measured in a certain unit, which may be meters, inches, furlongs, or whatever length unit we prefer to use. The result of this measurement can be written as

$$l = 3 \text{ m}. \tag{2.1}$$

The variable l has the physical dimension of length. In this chapter we write this as

$$l \sim [L]. \tag{2.2}$$

The square brackets are used in this chapter to indicate a physical dimension. The capital letter L denotes length, T denotes time, and M denotes mass. Other physical dimensions include electric charge and temperature. When dealing with physical dimensions two rules are useful.

Rule 1 When two variables are added, subtracted, or set equal to each other, they must have the same physical dimension.

To see the logic of this rule, consider the following example. Suppose we have an object with a length of 1 meter and a time interval of 1 second. This means that

$$\begin{aligned} l &= 1 \text{ m}, \\ t &= 1 \text{ s}. \end{aligned} \tag{2.3}$$

Since both variables have the same numerical value, we might be tempted to declare that

$$l = t. \tag{2.4}$$

It is, however, important to realize that the physical units that we use are arbitrary. Suppose, for example, that we had measured the length in feet rather than meters. In that case the measurements (2.3) would be given by

$$\begin{aligned} l &= 3 \text{ ft}, \\ t &= 1 \text{ s}. \end{aligned} \tag{2.5}$$

Now the numerical value of the same length measurement is different! Since the choice of the physical units is arbitrary, we can scale the relation between variables of different physical dimensions in an arbitrary way. For this reason these variables cannot be equal to each other. This implies that they cannot be added or subtracted either.

The first rule implies the following rule.

Rule 2 Mathematical functions, other than $f(\xi) = \xi^N$, can act on dimensionless numbers only.

To see this, let us consider as an example the function $f(\xi) = e^{\xi}$. Using a Taylor series (a topic we will discuss in Chapter 3), this function can be written as an expansion in powers ξ^n:

$$f(\xi) = 1 + \xi + \frac{1}{2}\xi^2 + \cdots \tag{2.6}$$

According to Rule 1 the different terms in this expression must have the same physical dimension. The first term (the number 1) is dimensionless; hence, all the other terms in the series must be dimensionless. This means that ξ must be a dimensionless number as well. This argument can be used for any function $f(\xi)$ whose Taylor expansion contains different powers of ξ. Note that the argument does not hold for a function that contains only a single power of ξ, such as $f(\xi) = \xi^2$. The argument does, however, hold for the function $f(\xi) = \xi + \xi^2$ because it consists of a superposition of different powers of ξ.

These rules have several applications in mathematical physics. Suppose we want to find the physical dimension of a force, as expressed in the basic dimensions of mass, length, and time. The only thing we need to do is take one equation that contains a force. In this case Newton's law $F = ma$ comes to mind. The mass m has physical dimension $[M]$, while the acceleration has dimension $[L/T^2]$. Rule 1 implies that force has the physical dimension $[ML/T^2]$.

Problem a The force F in a linear spring is related to the extension x of the spring by the relation $F = -kx$. Show that the spring constant k has dimension $[M/T^2]$.

Problem b The angular momentum $\mathbf{L}$ of a particle with momentum $\mathbf{p}$ at position $\mathbf{r}$ is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{2.7}$$

where $\times$ denotes the cross-product of two vectors. Show that angular momentum has the dimension $[ML^2/T]$.

Problem c A plane wave is given by the expression

$$u(\mathbf{r}, t) = e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{2.8}$$

where $\mathbf{r}$ is the position vector and t denotes time. Show that $\mathbf{k}$ has dimensions $[L^{-1}]$ and ω has dimensions $[T^{-1}]$.

In quantum mechanics, the behavior of a particle is characterized by a wave equation, called the Schrödinger equation. When the wave propagates along the x-axis as a function of time, i.e., the wave propagation is one-dimensional, this equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi, \tag{2.9}$$

where x denotes the position, t denotes the time, m the mass of the particle, and $V(x)$ the potential energy of the particle. At this point it is not clear what the wave function $\psi(x, t)$ is, and how this equation should be interpreted. Also, the meaning of the symbol $\hbar$ is not yet defined. We can, however, determine the physical dimension of $\hbar$ without knowing the meaning of this variable.

Problem d Compare the physical dimensions of the left-hand side of equation (2.9) with the first term on the right-hand side and show that the variable $\hbar$ has the physical dimension of angular momentum. You can use Problem b to show this.

Chapter 8 has more on this variable, which turns out to be the reduced Planck's constant in the Schrödinger equation.

When using dimensional arguments, one needs to be aware of a curious property of the logarithm. Suppose L_1 and L_2 are both lengths, then the ratio L_1/L_2 is dimensionless. We can let the logarithm act on this ratio and use the following property of the logarithm

$$\ln(L_1/L_2) = \ln(L_1) - \ln(L_2). \tag{2.10}$$

This equation is mathematically correct, but note that in the left-hand side the logarithm acts on a dimensionless function, while in the right-hand side it acts on a length! This means that Rule 2 can be relaxed for the logarithm. Doing so, however, can be a source of confusion and mistakes, and it is best to let all mathematical functions act on dimensionless variables only.

2.2 A trick for finding mistakes

The requirement that all terms in an equation have the same physical dimension is an important tool for spotting mistakes. Cipra (2000) gives many useful tips for spotting errors in his delightful book *Misteaks* [sic] ... *and How to Find Them Before the Teacher Does*. As an example of using dimensional analysis for spotting mistakes, we consider the erroneous equation

$$E = mc^3, \tag{2.11}$$

where E denotes energy, m denotes mass, and c is the speed of light. Let us first find the physical dimension of energy. The work done by a force $\mathbf{F}$ over a displacement $d\mathbf{r}$ is given by $dE = \mathbf{F}\cdot d\mathbf{r}$. We showed in Section 2.1 that force has the dimension $[ML/T^2]$. This means that energy has the dimension $[ML^2/T^2]$. The speed of light has dimension $[L/T]$, which means that the right-hand side of equation (2.11) has physical dimension $[ML^3/T^3]$. This is not equal to the dimensions of energy on the left-hand side. Therefore, expression (2.11) is wrong.

Problem a Now that we have determined that expression (2.11) is incorrect, we can use the requirement that the dimensions of the different terms must match to guess how to set it right. Show that the right-hand side must be divided by a velocity to match the dimensions.

It is not clear that the right-hand side must be divided by the speed of light to give the correct expression $E = mc^2$. Dimensional analysis tells us only that it must be divided by something with the dimension of velocity. For all we know, that could be the speed at which the average snail moves.

Problem b Is the following equation dimensionally correct?

$$(\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p. \tag{2.12}$$

In this expression $\mathbf{v}$ is the velocity of fluid flow, p is the pressure, and ∇ is the gradient vector (which essentially is a derivative with respect to the space coordinates, as explained in Chapter 5). To find the dimension of the right-hand side, you can use the fact that pressure is force per unit area.

Problem c Answer the same question for the expression that relates the particle velocity v to the pressure p in an acoustic medium:

$$v = \frac{p}{\rho c}, \tag{2.13}$$

where ρ is the mass density and c is the velocity of propagation of acoustic waves.

Problem d In quantum mechanics, the energy E of the harmonic oscillator is given by

$$E_n = \hbar\omega\,(n + 1/2), \tag{2.14}$$

where ω is a frequency, n is a dimensionless integer, and $\hbar$ is Planck's constant divided by 2π, as introduced in Problem d of the previous section. Verify if this expression is dimensionally correct.

In general, it is a good idea to carry out a dimensional analysis while working in mathematical physics. This may help in finding the mistakes that we all make while doing derivations. It takes a little while to become familiar with the dimensions of properties that are used most often, but this is an investment that pays off in the long run.

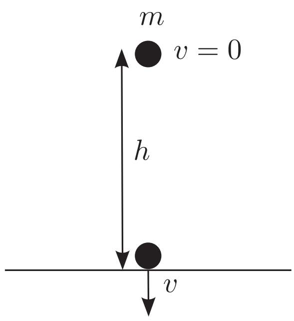

Figure 2.1 Definition of the variables for a falling ball.

2.3 Buckingham pi theorem

In this section we introduce the Buckingham pi theorem. This theorem can be used to find the relation between physical parameters, based on dimensional arguments. As an example, let us consider a ball shown in Figure 2.1 with mass m that is dropped from a height h. We want to find the velocity with which it strikes the ground. The potential energy of the ball before it is dropped is mgh, where g is the acceleration of gravity. This energy is converted into kinetic energy $\frac{1}{2}mv^2$ as it strikes the ground. Equating these quantities and solving for the velocity gives

$$v = \sqrt{2gh}. \tag{2.15}$$

Let us suppose we did not know about classical mechanics. In that case, dimensional analysis could be used to guess relation (2.15). We know that the velocity is some function of the acceleration of gravity, the initial height, and the mass of the particle: $v = f(g, h, m)$. The physical dimensions of these properties are given by

$$v \sim [L/T], \quad g \sim [L/T^2], \quad h \sim [L], \quad m \sim [M]. \tag{2.16}$$

Let us consider the dimension mass first. The dimension mass enters only the variable m. We cannot combine the variable m with the parameters g and h in any way to arrive at a quantity that is independent of mass. Therefore, the velocity does not depend on the mass of the particle. Next we consider the dimension time. The velocity depends on time as $[T^{-1}]$, the acceleration of gravity as $[T^{-2}]$, and h is independent of time. This means that we can match the dimension time only when

$$v \sim \sqrt{g}. \tag{2.17}$$

In this expression the left-hand side depends on the length as $[L]$, while the right-hand side varies with length as $[L^{1/2}]$. We have, however, not used the height h yet. The dimension length can be made to match if we multiply the right-hand side with $\sqrt{h}$. This means that the only combination of g and h that gives a velocity is given by

$$v \sim \sqrt{gh}. \tag{2.18}$$

This result agrees with expression (2.15), which was derived using classical mechanics. Note that in order to arrive at expression (2.18) we used only dimensional arguments, and did not need to have any knowledge from classical mechanics other than the fact that the velocity depends only on g and h. The dimensional analysis that led to expression (2.18), however, does not tell us what the proportionality constant in that expression is. Because the proportionality constant is dimensionless, it cannot be found by dimensional analysis.

The treatment given here may appear to be cumbersome. This analysis, however, can be carried out in a systematic fashion using the *Buckingham pi theorem* (Buckingham, 1914):

Buckingham pi theorem If a problem contains N variables that depend on P physical dimensions, then there are $N - P$ dimensionless numbers that describe the physics of the problem.

The original paper of Buckingham is very clear, but as we will see at the end of this section, this theorem is not foolproof. Let us first apply the theorem to the problem of the falling ball. We have four variables: v, g, h, and m, so that $N = 4$. These variables depend on the physical dimensions $[M]$, $[L]$, and $[T]$, hence $P = 3$. According to the Buckingham pi theorem, $N - P = 1$ dimensionless number characterizes the problem. We want to express the velocity in the other parameters; hence, we seek a dimensionless number of the form

$$vg^{\alpha}h^{\beta}m^{\gamma} \sim [1], \tag{2.19}$$

where the notation in the right-hand side means that it is dimensionless. Let us seek the exponents α, β, and γ that make the left-hand side dimensionless. Inserting the dimensions of the different variables then gives the following dimensions

$$\left[\frac{L}{T}\right]\left[\frac{L^{\alpha}}{T^{2\alpha}}\right]\left[L^{\beta}\right]\left[M^{\gamma}\right] \sim [1]\,. \tag{2.20}$$

The left-hand side depends on length as $[L^{1+\alpha+\beta}]$. The left-hand side can only be independent of length when the exponent is equal to zero. Applying the same reasoning to each of the dimensions length, time, and mass, then gives

$$\begin{array}{ll} \text{dimension } [L]: & 1+\alpha+\beta = 0, \\ \text{dimension } [T]: & -1-2\alpha = 0, \\ \text{dimension } [M]: & \gamma = 0. \end{array} \tag{2.21}$$

This constitutes a system of three equations with three unknowns.